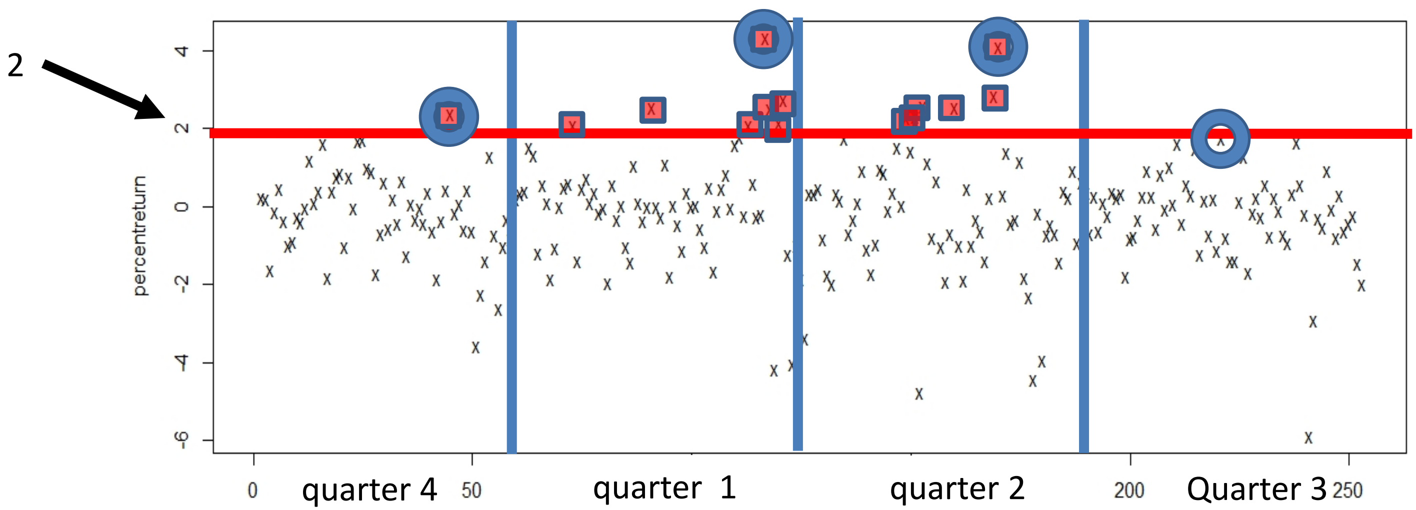
Financial Risk
4-th quarter 2022/23
Lecture2: Peaks over thresholds

The big recession 2009

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

Gudrun January 2005
326 MEuro loss
72 % due to forest losses
4 times larger than second largest

Apple losses (= - $100 \times \frac{\text{price tomorrow} - \text{price today}}{\text{price today}}$) one year back

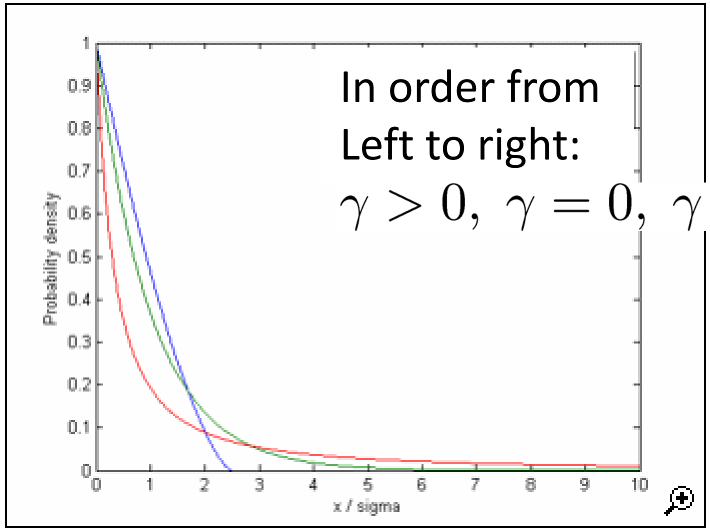


Maximum quarterly loss ■ excess of the level $u = 2$

How large is the risk of a big quarterly loss? BM

How large is the risk of a big loss tomorrow? PoT

The GP distribution: $H(x) = 1 - \left(1 + \frac{\gamma}{\sigma} x\right)_+^{-1/\gamma}$



density function of Generalized Pareto distribution

$$h(x) = \frac{d}{dx} H(x) = \frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma} x\right)_+^{-1/\gamma - 1} \quad \left(= \frac{1}{\sigma} e^{-x/\sigma} \ \text{ if } \ \gamma = 0\right)$$

$\gamma \geq 0$ the distribution has left endpoint *0* and right endpoint ∞

$\gamma < 0$ the distribution has left endpoint *0* and right endpoint $\sigma/|\gamma|$

the distribution is “heavytailed” for $\gamma > 0$. Then moments of order greater than $1/\gamma$ are infinite/don’t exist, exactly as for the Generalized Extreme Value distribution

- Peaks over thresholds (PoT) method *(Coles p. 74-91)*

- Choose (high) threshold u and from i.i.d observations $Y_1, \dots, Y_n \sim F$ obtain N threshold excesses $X_1 = Y_{t_1} - u, \dots, X_N = Y_{t_N} - u$, where $t_1, \dots, t_N$ are the times of threshold exceedance
- Assume $X_1, \dots, X_N$ are i.i.d and GP distributed and that $t_1, \dots, t_N$ are the occurrence times of an independent Poisson process, so that N has a Poisson distribution
- Use $X_1, \dots, X_n$ to estimate the GP parameters and N to estimate the mean of the Poisson distribution
- Estimate tail $\bar{F}(x) = 1 - F(x) = \bar{F}(u)\bar{F}_u(x-u)$, where $\bar{F}_u(x-u)$ = the conditional distribution function of threshold excesses, by

$$\hat{\bar{F}}(x) = \frac{N}{n}\,\hat{\bar{F}}_u(x-u)$$

Details:

Assume the random variable Y has d.f. F and let u be a (high) level. The distribution of exceedances then is the conditional distribution of Y$-u$ given that Y is larger than u, *i.e.* it has d.f.

$$F_u(x) = P(Y - u \leq x | Y > u) = \frac{P(Y \leq x + u \text{ and } Y > u)}{P(Y > u)}$$

$$= \frac{F(x+u) - F(u)}{1 - F(u)}$$

(and hence $\bar{F}_u(x) = 1 - F_u(x) = 1 - \frac{F(x+u)-F(u)}{1-F(u)}$

$$= \frac{1-F(x+u)}{1-F(u)} = \frac{\bar{F}(x+u)}{\bar{F}(u)}).$$

Exercise: Show that if *F(x)* is a GP distribution, then also $F_u(x)$ is a GP distribution, and express the parameters of $F_u(x)$ in terms of the parameters of $F(x)$ (Treat $\gamma \neq 0$ and $\gamma = 0$ separately.)

More details

N = the (random) number of exceedances of the threshold u by $Y_1, \dots, Y_n$. The ratio N$/n$ is a natural estimator of $\bar{F}(u)$. Assume we have computed estimators $\hat{\sigma}, \hat{\gamma}$ of the parameters in the GP distribution $\bar{F}_u(x) = H(x)$. Since $\bar{F}(x) = \bar{F}(u)\bar{F}_u(x-u)$, a natural estimator of the "tail function" $\bar{F}(x)$, for $x > u$, then is

$$\hat{\bar{F}}(x) = \frac{N}{n}\,\hat{\bar{H}}(x-u) = \frac{N}{n}\left(1 + \frac{\hat{\gamma}}{\hat{\sigma}}(x-u)\right)^{-1/\hat{\gamma}}$$

Solving $1 - \hat{\bar{F}}\left(x_p\right) = p$ for x_p we get an estimator of the p-th quantile of X:

$$x_p = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{n}{N}(1-p)\right)^{-\hat{\gamma}} - 1\right), \qquad \text{for } p > 1 - \frac{N}{n}$$

(Why not just estimate $\bar{F}(x)$ by $N(x)/n$? Because if x is a very high level then $N(x)$ is small or zero, and then this estimator is useless -- and it is such very large x-es we are interested in.)

The Poisson process

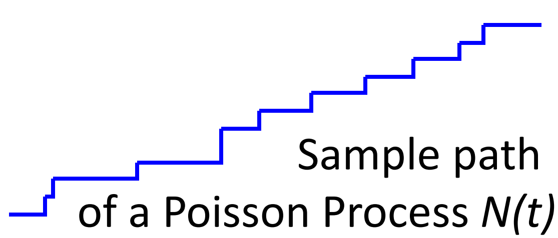
Model for times of occurrence of events which occur "randomly" in time, with a constant "intensity", e.g, radioactive decay, times when calls arrive to a telephone exchange, times when traffic accidents occur ... Can be mathematically described as a counting process *N(t) = #events in [0, t].* The counting process *N(t)* is a Poisson process if

a) The numbers of events which occur in disjoint time intervals are mutually independent

b) *N(s+t) – N(s)* has a Poisson distribution for any *s, t ≥ 0, i.e.*

$$P\big(N(s+t) - N(s) = k\big) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad \text{for any} \quad s, t \geq 0, \; k = 1, 2, \ldots$$

λ is the "intensity" parameter. It is the expected number of events in any time interval of length *1.*



Sample path of a Poisson Process *N(t)*

A connection between the PoT and Block Maxima methods

Suppose the PoT model holds, so values larger than u occur as a Poisson process with intensity λ; this process is independent of the sizes of the excesses; these are i.i.d. and have a GP distribution $H(x) = 1 - \left(1 + \frac{\gamma}{\sigma}x\right)_+^{-1/\gamma}$. $M_T =$ the maximum in the time interval $[0, T]$. Then if $x > 0$

$$
\begin{aligned}
P\big(M_T \le u + x\big) &= \sum_{k=0}^{\infty} P\big(M_T \le u + x, \text{ there are k exceedances in [0, T]}\big) \\
&= \sum_{k=0}^{\infty} H(x)^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \sum_{k=0}^{\infty} (1 - (1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma})^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \exp\{(1 - (1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma})\lambda T\} \exp\{-\lambda T\} \\
&= \exp\{-(1 + \frac{\gamma}{\sigma}x)_+^{-1/\gamma} \lambda T\} \\
&= \exp\{-(1 + \gamma \frac{x - ((\lambda T)^\gamma - 1)\sigma/\gamma}{\sigma(\lambda T)^\gamma})_+^{-1/\gamma}\}
\end{aligned}
$$

Dependence

Often excesses of u are not independent, and then the formula on the previous slide is not valid -- however it still works if one ”declusters”, and instead of excesses uses cluster maxima. Declustering is discussed later in the course, in Lecture 5, p. 11 - 12