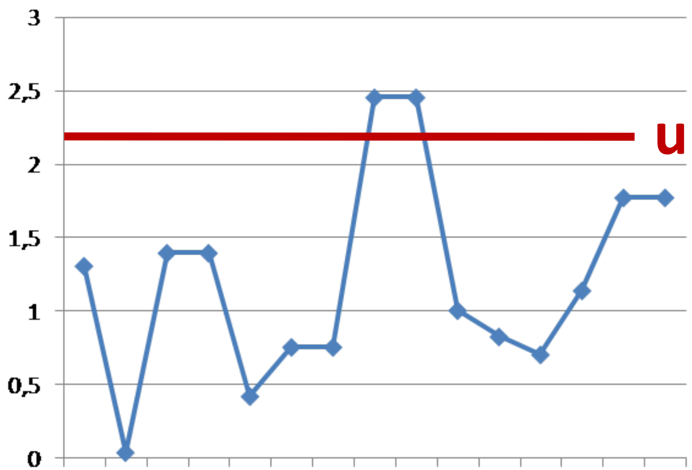
Financial Risk
4-th quarter 2023
Lecture 4: Dependence, backtesting

"As an alternative to the traditional 30-year mortgage, we also offer an interest-only mortgage, balloon mortgage, reverse mortgage, upside down mortgage, inside out mortgage, loop-de-loop mortgage, and the spinning double axel mortgage with a triple lutz."

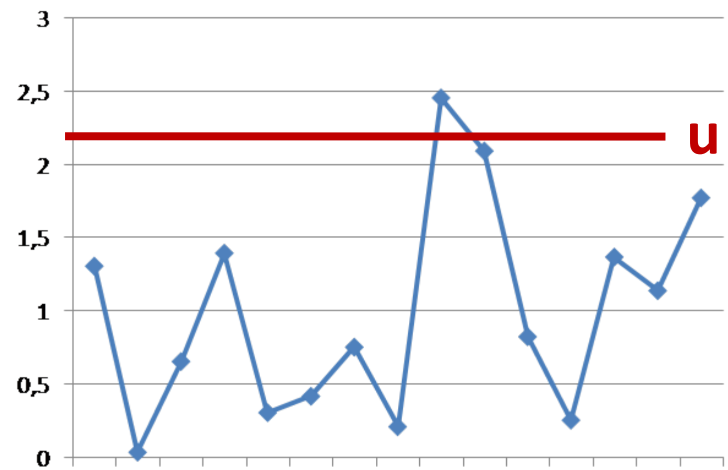
Gudrun January 2005
326 MEuro loss
72 % due to forest losses
4 times larger than second largest

Dependence: Extreme Value Statistics for stationary time series

(Coles p. 92-104)



$X_1, X_2, \ldots$ stationary, d.f. *F(x)*



$\tilde{X}_1, \tilde{X}_2, \ldots$ i.i.d., same d.f. *F(x)*
("associated i.i.d. sequence")

Dependence → extremes typically come in small "clusters"

θ = "***Extremal index***" = 1/"asymptotic mean cluster length"

- typically $P\big(M_n \le x\big) \approx F(x)^{\theta n}$ for n large
- typically clusters asymptotically i.i.d., dependence within clusters
- typically tail of cluster maxima asymptotically same as $\bar{F}(x)$ **!!**
- typically the GEV distributions the only possible limit distributions

The block maxima method for stationary time series

If blocks are sufficiently long, then block maxima (typically) are approximately independent, and one can use Extreme Value Statistics in precisely the same way as for i.i.d. sequences

The PoT method for stationary time series

1. Decluster: identify approximately i.i.d clusters of large values by

a) Block method: divide observations up into blocks of a fixed length *r*, all values in a block which exceed the level *u* is a cluster

b) Blocks-runs method: the first cluster starts at first exceedance of *u* and contains all excesses of *u* within a fixed length *r* thereafter. The second cluster starts at the next exceedance of *u* and contains all excesses of *u* within *r* thereafter, and so on. . .

c) Runs method: the first cluster starts with the first exceedance of *u* and stops as soon as there is a value below *u*, the second cluster starts with the next exceedance of *u*, and so on …

2. $\hat{\theta} = \dfrac{\text{no. of clusters}}{\text{no. of exceedances}}$ estimate of the *extremal index*

3. ***PoT:*** Use standard i.i.d. PoT model, but with excesses replaced by cluster maxima, and excedance times replaced by the times when cluster maxima occur. (A bit of a miracle this works. Proof not given here.)

4. Use $P(M_n \leq x) \approx F(x)^{\theta n}$ to switch between block maxima and PoT **or**

5. Use formula for i.i.d. variables with excesses replaced by excesses by cluster maxima and the number of excesses replaced by the number of clusters

The i.i.d. formula: Suppose excesses are GP distributed and occur as a Poisson process which is independent of the sizes of excesses. Let M_T be the maximum in the interval $[0, T]$ and $x > 0$. Then

$$
\begin{aligned}
P\big(M_T \leq u + x\big) &= \sum_{k=0}^{\infty} P\big(M_T \leq u + x,\ \text{there are k exceedances in [0, T]}\big) \\
&= \sum_{k=0}^{\infty} H(x)^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \sum_{k=0}^{\infty} (1 - (1 + \frac{\gamma}{\sigma} x)_+^{-1/\gamma})^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \exp\{(1 - (1 + \frac{\gamma}{\sigma} x)_+^{-1/\gamma})\lambda T\} \exp\{-\lambda T\} \\
&= \exp\{-(1 + \frac{\gamma}{\sigma} x)_+^{-1/\gamma} \lambda T\} \\
&= \exp\{-(1 + \gamma \frac{x - ((\lambda T)^{\gamma} - 1)\sigma/\gamma}{\sigma(\lambda T)^{\gamma}})_+^{-1/\gamma}\}
\end{aligned}
$$

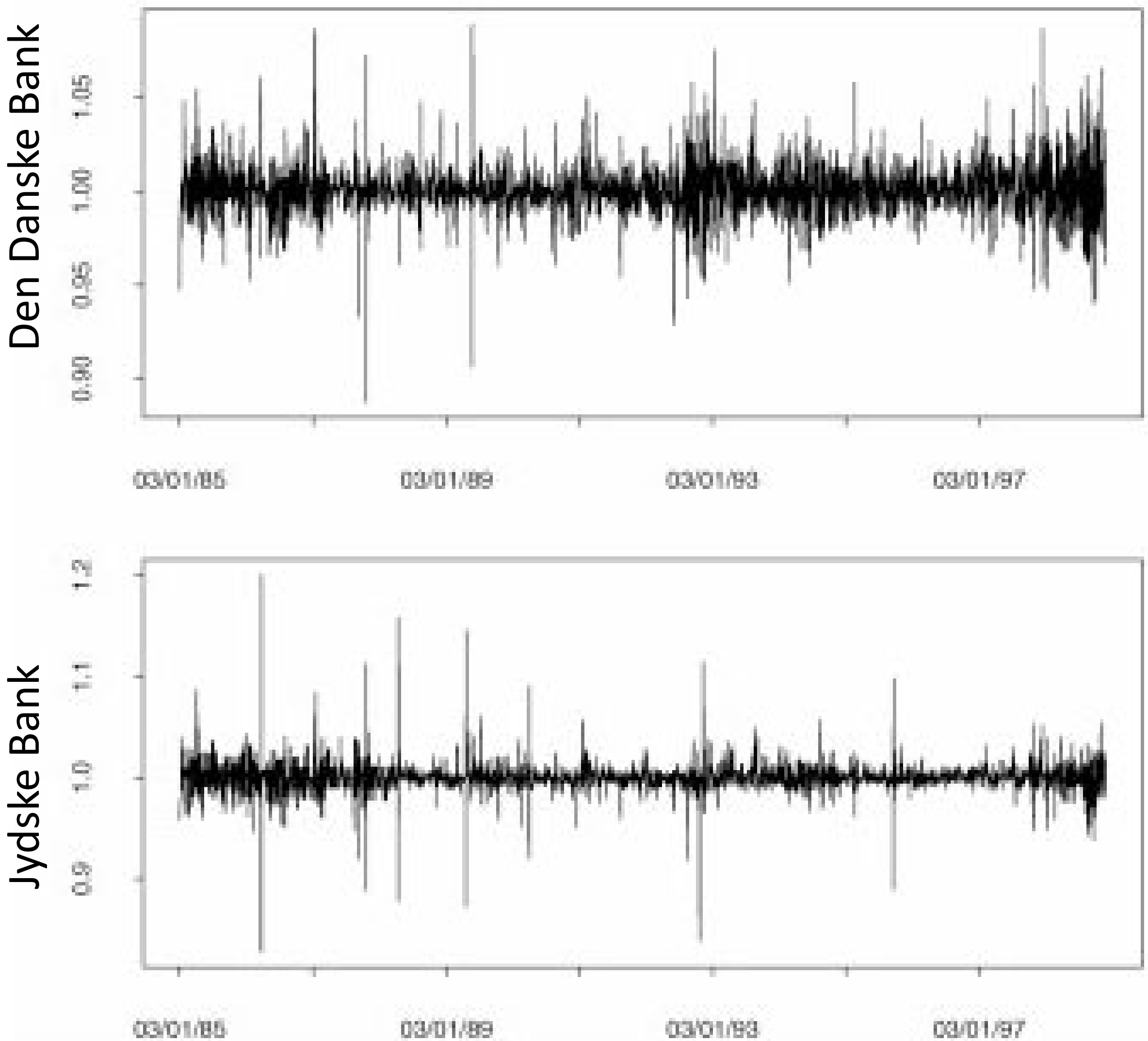
Estimating value at risk by extreme value methods;

(Sarah Lauridsen, Extremes 3, 107-144, 2000)

VaR = high quantiles of the loss-profits distribution

- empirical quantiles
- unconditional Gaussian method
- conditional Gaussian method
- GEV + different extremal index estimators
- GP pretending independence
- GP with declustering
- GARCH + GP residuals, conditional
- GARCH + GP residuals, unconditional

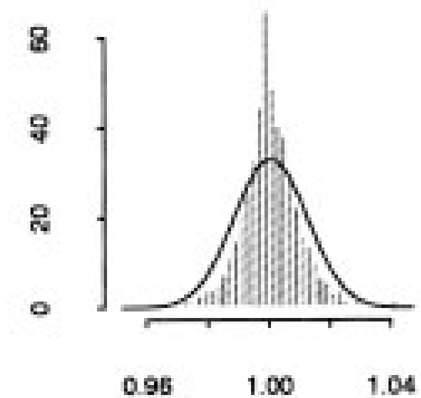
Compared, and evaluated via backtesting



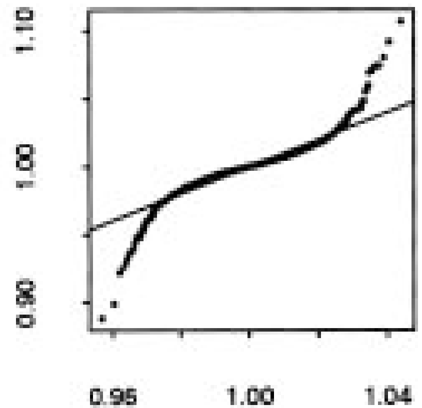
Daily returns from Jan. 1, 1985 to Nov. 27, 1998

Synthetic portfolio: 50 MDKK Danske Bank + 50 MDKK Jydske Bank

Empirical and Normal



histogram with estimated normal density
(13 left values and 10 right values not shown)



normal qq-plot

VaR in mDKr estimated by Gaussian and empirical method

1-day VaR	95%	96%	97%	98%	99%	99.9%	99.99%
Gaussian method	−1.93	−2.05	−2.21	−2.42	−2.75	−3.67	−4.42
Empirical method	−1.66	−1.85	−2.07	−2.43	−3.10	−7.55	—

To assume returns normally distributed and i.i.d. gives easy calculations, also for complex portfolios

-- but, distribution doesn't fit in the tails, independence not OK

-- the empirical method gives no estimates for extreme quantiles

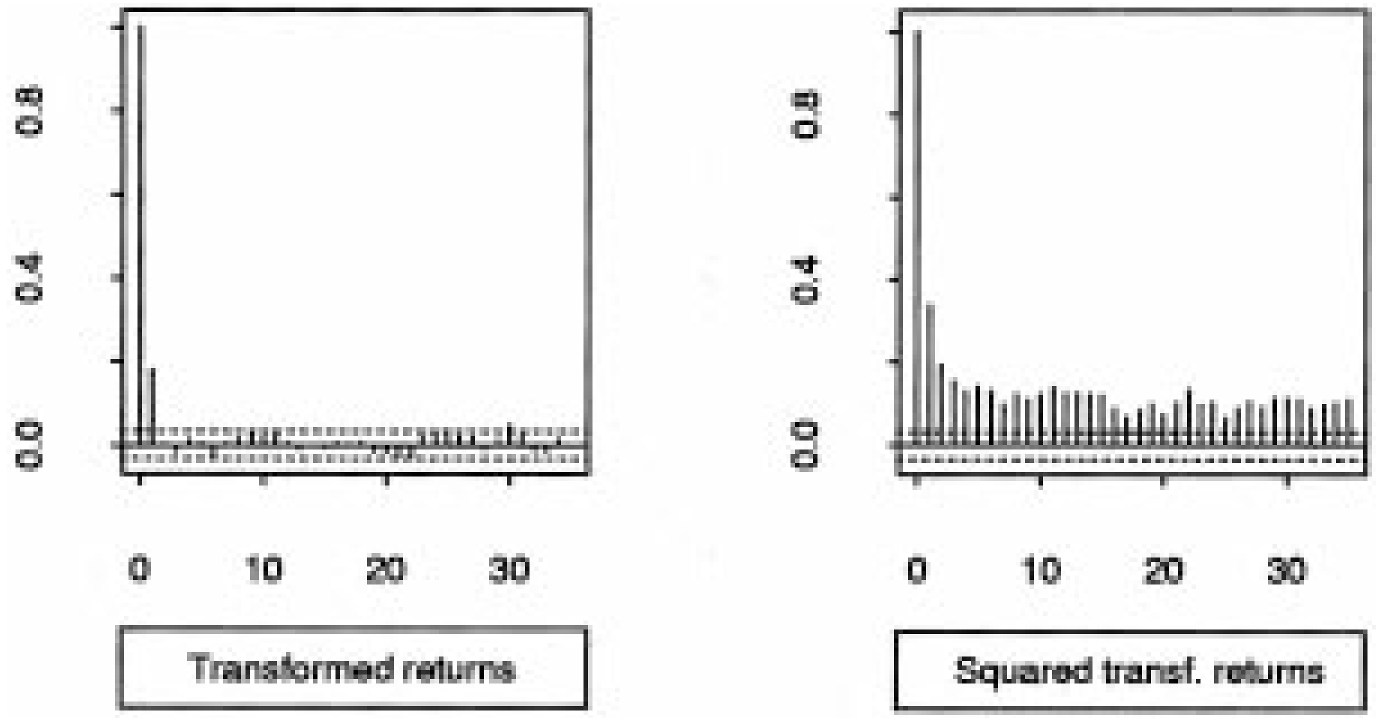
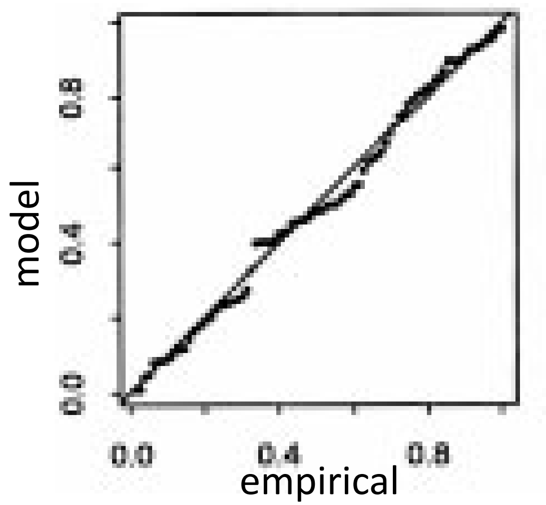


Figure 3. Auto correlograms for the series of daily returns on the bank portfolio. The returns have been transformed to have Gaussian marginals.

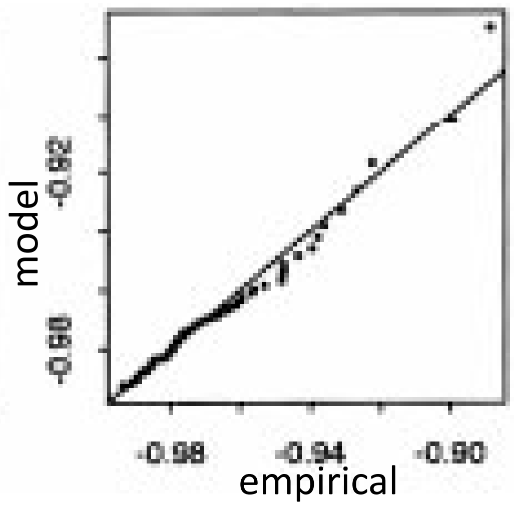
checked dependence by transforming to normal marginal distribution and computing correlations → clear and strong dependence for squared returns.

Block Maxima for 42 days approximately independent (figure not shown)

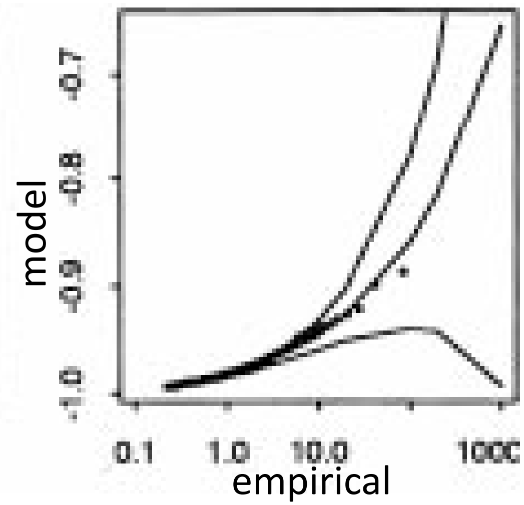
Block Maxima



pp-plot against GEV,
42 day Block Maxima



qq-plot against GEV,
42 day Block Maxima



return level plot assuming GEV,
42 day Block Maxima

GEV distribution fits the data well, and 42 days maxima interesting for firm survival, but how can one get from there to overnight VaR?

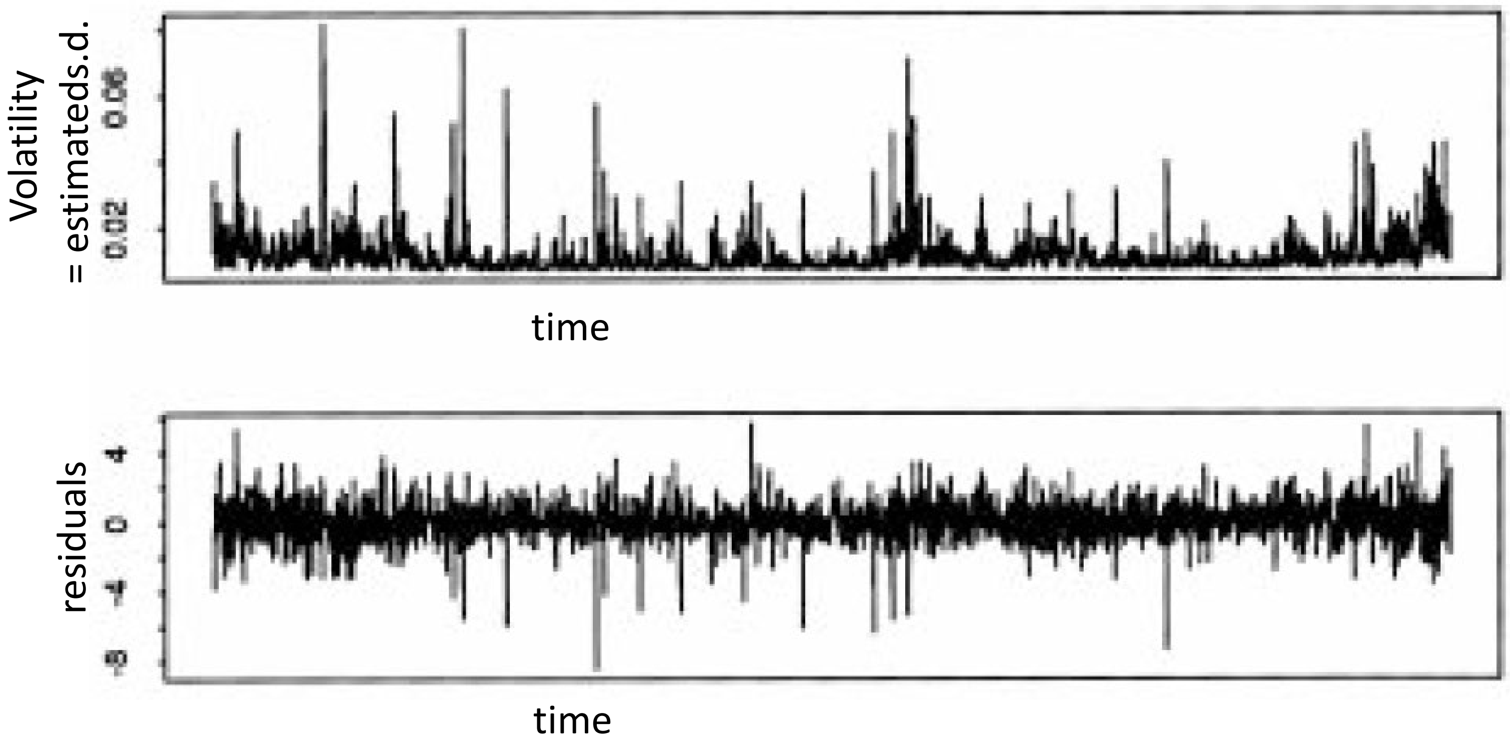
α- quantile of overnight P&L-distribution estimated by $\alpha^{n\theta}$- quantile of n-day maxima (prove this!)

- But θ difficult to estimate

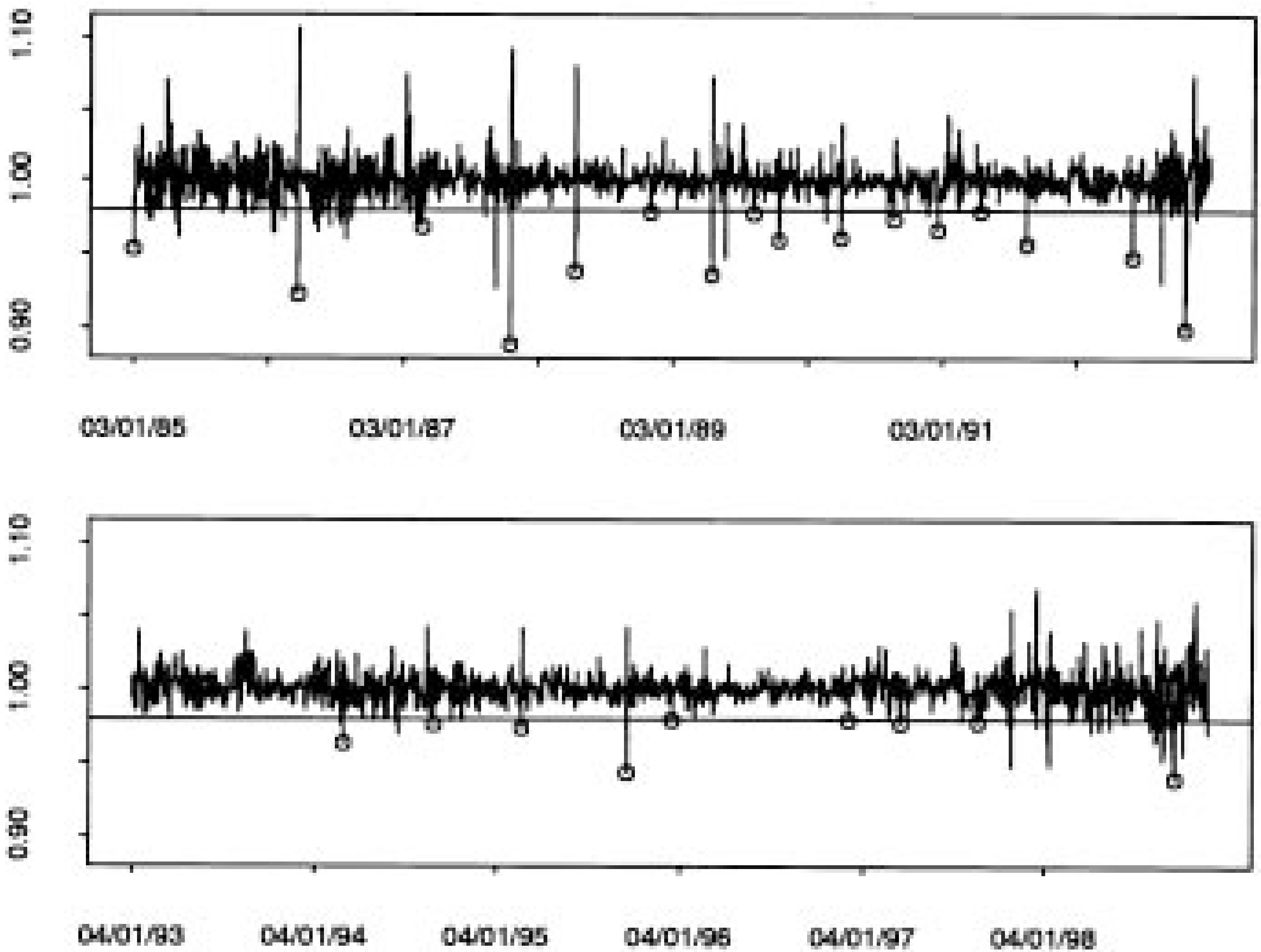
Garch

fit Garch model, compute residuals, fit GP distribution to residuals, compute quantiles of the resulting estimated distribution of returns (computation done by simulation).

Can be done ***conditionally***, using volatility today to compute quantiles for the portfolio tomorrow or ***unconditionally*** – for longrun behaviour of portfolio



PoT



Cluster minima: level *u = 0.98,* separation length *r = 40*

Backtesting

- compute VaR from the first six years of data, see if it "is violated", i.e. if next days return is lower than VaR, repeat again using six years of data but starting one day later, two days later, … count number of violations

- expected no. of violations in parentheses (appr 2000 observations)

Backtesting results, violations of 1-day VaR

Empirical method

	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	119 (99.3)	100 (79.4)	74 (59.6)	56 (39.7)	30 (19.9)	1 (2.0)	– (0.2)
S&P 500	101 (159.9)	77 (127.9)	55 (95.9)	35 (64.0)	20 (32.0)	3 (3.2)	– (0.3)
B&O	90 (99.3)	72 (79.4)	53 (59.6)	35 (39.7)	19 (19.9)	0 (2.0)	– (0.2)
Carlsberg	81 (99.3)	68 (79.4)	56 (59.6)	39 (39.7)	17 (19.9)	0 (2.0)	– (0.2)
DS 1912	117 (99.3)	93 (79.4)	73 (59.6)	52 (39.7)	27 (19.9)	3 (2.0)	– (0.2)
ISS	148 (99.3)	126 (79.4)	93 (59.6)	63 (39.7)	23 (19.9)	1 (2.0)	– (0.2)
Novo B	113 (99.3)	90 (79.4)	70 (59.6)	49 (39.7)	26 (19.9)	2 (2.0)	– (0.2)
Svendborg	122 (99.3)	96 (79.4)	77 (59.6)	55 (39.7)	30 (19.9)	4 (2.0)	– (0.2)

Unconditional Gaussian method

	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	76 (99.3)	66 (79.4)	62 (59.6)	47 (39.7)	30 (19.9)	16 (2.0)	10 (0.2)
S&P 500	101 (159.9)	79 (127.9)	59 (95.9)	36 (64.0)	22 (32.0)	5 (3.2)	3 (0.3)
B&O	61 (99.3)	56 (79.4)	49 (59.6)	40 (39.7)	28 (19.9)	13 (2.0)	9 (0.2)
Carlsberg	63 (99.3)	55 (79.4)	48 (59.6)	39 (39.7)	27 (19.9)	11 (2.0)	5 (0.2)
DS 1912	105 (99.3)	95 (79.4)	85 (59.6)	67 (39.7)	48 (19.9)	19 (2.0)	8 (0.2)
ISS	81 (99.3)	67 (79.4)	57 (59.6)	41 (39.7)	29 (19.9)	19 (2.0)	11 (0.2)
Novo B	88 (99.3)	72 (79.4)	61 (59.6)	54 (39.7)	41 (19.9)	16 (2.0)	7 (0.2)
Svendborg	108 (99.3)	98 (79.4)	84 (59.6)	73 (39.7)	57 (19.9)	22 (2.0)	9 (0.2)

Conditional Gaussian

	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	77 (99.3)	64 (79.4)	51 (59.6)	41 (39.7)	27 (19.9)	10 (2.0)	6 (0.2)
S&P 500	151 (159.9)	127 (127.9)	88 (95.9)	61 (64.0)	33 (32.0)	7 (3.2)	4 (0.3)
B&O	71 (99.3)	61 (79.4)	53 (59.6)	42 (39.7)	27 (19.9)	11 (2.0)	7 (0.2)
Carlsberg	60 (99.3)	53 (79.4)	44 (59.6)	36 (39.7)	26 (19.9)	11 (2.0)	5 (0.2)
DS 1912	90 (99.3)	71 (79.4)	57 (59.6)	45 (39.7)	23 (19.9)	9 (2.0)	4 (0.2)
ISS	90 (98.2)	80 (78.6)	70 (58.9)	53 (39.3)	37 (19.6)	18 (2.0)	14 (0.2)
Novo B	72 (99.3)	61 (79.4)	46 (59.6)	37 (39.7)	24 (19.9)	8 (2.0)	3 (0.2)
Svendborg	97 (99.3)	87 (79.4)	72 (59.6)	52 (39.7)	31 (19.9)	10 (2.0)	4 (0.2)

GEV and simple blocks estimator (95% threshold)

	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	71 (99.3)	57 (79.4)	36 (59.6)	19 (39.7)	11 (19.9)	0 (2.0)	0 (0.2)
S&P	43 (159.9)	37 (127.9)	25 (95.9)	15 (64.0)	8 (32.0)	2 (3.2)	1 (0.3)
B&O	54 (99.3)	46 (79.4)	34 (59.6)	23 (39.7)	8 (19.9)	0 (2.0)	0 (0.2)
Carlsberg	40 (99.3)	33 (79.4)	24 (59.6)	16 (39.7)	7 (19.9)	0 (2.0)	0 (0.2)
DS 1912	66 (99.3)	55 (79.4)	44 (59.6)	28 (39.7)	14 (19.9)	1 (2.0)	0 (0.2)
ISS	79 (99.1)	60 (79.3)	36 (59.5)	20 (39.6)	11 (19.8)	0 (2.0)	0 (0.2)
Novo B	62 (99.3)	54 (79.4)	40 (59.6)	26 (39.7)	11 (19.9)	1 (2.0)	0 (0.2)
Svendborg	77 (99.3)	67 (79.4)	54 (59.6)	34 (39.7)	12 (19.9)	1 (2.0)	0 (0.2)

Backtesting results, violations of 1-day VaR GEV and simple blocks estimator (99% threshold)							
	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	140 (99.3)	106 (79.4)	78 (59.6)	56 (39.7)	17 (19.9)	1 (2.0)	0 (0.2)
S&P500	79 (159.9)	56 (127.9)	36 (95.9)	25 (64.0)	13 (32.0)	2 (3.2)	2 (0.3)
B&O	80 (99.3)	63 (79.4)	51 (59.6)	38 (39.7)	15 (19.9)	0 (2.0)	0 (0.2)
Carlsberg	80 (99.3)	70 (79.4)	51 (59.6)	33 (39.7)	16 (19.9)	0 (2.0)	0 (0.2)
DS 1912	108 (99.3)	94 (79.4)	75 (59.6)	48 (39.7)	23 (19.9)	4 (2.0)	0 (0.2)
ISS	157 (99.1)	129 (79.3)	97 (59.5)	61 (39.6)	23 (19.8)	2 (2.0)	0 (0.2)
Novo B	115 (99.3)	93 (79.4)	71 (59.6)	49 (39.7)	25 (19.9)	2 (2.0)	0 (0.2)
Svendborg	125 (99.3)	106 (79.4)	82 (59.6)	60 (39.7)	26 (19.9)	3 (2.0)	0 (0.2)
GEV and blocks estimator (95% threshold)							
	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	139 (99.3)	107 (79.4)	78 (59.6)	54 (39.7)	18 (19.9)	1 (2.0)	0 (0.2)
S&P500	97 (159.9)	75 (127.9)	57 (95.9)	34 (64.0)	15 (32.0)	3 (3.2)	2 (0.3)
B&O	89 (99.3)	73 (79.4)	58 (59.6)	44 (39.7)	17 (19.9)	1 (2.0)	0 (0.2)
Carlsberg	77 (99.3)	65 (79.4)	46 (59.6)	33 (39.7)	14 (19.9)	0 (2.0)	0 (0.2)
DS 1912	112 (99.3)	90 (79.4)	70 (59.6)	49 (39.7)	21 (19.9)	3 (2.0)	0 (0.2)
ISS	147 (99.1)	129 (79.3)	94 (59.5)	46 (39.6)	20 (19.8)	1 (2.0)	0 (0.2)
Novo B	109 (99.3)	88 (79.4)	65 (59.6)	45 (39.7)	21 (19.9)	2 (2.0)	0 (0.2)
Svendborg	118 (99.3)	100 (79.4)	80 (59.6)	58 (39.7)	29 (19.9)	2 (2.0)	0 (0.2)
GEV and blocks estimator (99% threshold)							
	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	166 (99.3)	133 (79.4)	99 (59.6)	62 (39.7)	23 (19.9)	1 (2.0)	0 (0.2)
S&P500	88 (159.9)	66 (127.9)	45 (95.9)	31 (64.0)	14 (32.0)	3 (3.2)	2 (0.3)
B&O	86 (99.3)	75 (79.4)	59 (59.6)	45 (39.7)	18 (19.9)	1 (2.0)	0 (0.2)
Carlsberg	102 (99.3)	78 (79.4)	61 (59.6)	42 (39.7)	20 (19.9)	0 (2.0)	0 (0.2)
DS 1912	126 (99.3)	104 (79.4)	87 (59.6)	55 (39.7)	27 (19.9)	5 (2.0)	1 (0.2)
ISS	187 (99.1)	156 (79.3)	114 (59.5)	75 (39.6)	27 (19.8)	2 (2.0)	0 (0.2)
Novo B	133 (99.3)	109 (79.4)	81 (59.6)	58 (39.7)	30 (19.9)	4 (2.0)	0 (0.2)
Svendborg	144 (99.3)	121 (79.4)	99 (59.6)				
GPD							
	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	118 (99.3)	98 (79.4)	78 (59.6)	57 (39.7)	20 (19.9)	1 (2.0)	0 (0.2)
S&P500	26 (20.8)	26 (16.6)	21 (12.5)	16 (8.32)	5 (4.1)	2 (0.4)	1 (0.04)
B&O	89 (98.9)	74 (79.1)	54 (59.3)	38 (39.5)	16 (19.8)	2 (2.0)	0 (0.2)
Carlsberg	69 (95.6)	56 (76.8)	46 (57.6)	33 (38.4)	16 (19.2)	1 (2.0)	0 (0.2)
DS 1912	98 (74.3)	76 (59.4)	65 (44.6)	41 (29.7)	24 (14.9)	4 (1.5)	1 (0.1)
ISS	151 (99.3)	128 (79.4)	95 (59.6)	61 (39.7)	26 (19.9)	3 (2.0)	0 (0.2)
Novo B	116 (99.2)	89 (79.4)	70 (59.5)	50 (39.7)	27 (19.8)	4 (2.0)	1 (0.2)
Svendborg	110 (88.9)	96 (71.1)	68 (53.3)	52 (35.5)	25 (17.8)	4 (1.8)	0 (0.2)

Backtesting

- compute VaR from the first six years of data, see if it “is violated”, i.e. if next days return is lower than VaR, repeat again using six years of data but starting one day later, two days later, … count number of violations

- expected no. of violations in parentheses

Backtesting

Backtesting results, violations of 1-day VaR
GARCH based extreme value method, conditional

	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	109 (99.3)	91 (79.4)	64 (59.6)	44 (39.7)	18 (19.9)	1 (2.0)	0 (0.2)
S&P 500	145 (157.8)	123 (126.2)	88 (94.7)	60 (63.1)	34 (31.6)	6 (3.2)	2 (0.3)
B&O	95 (99.3)	73 (79.4)	53 (59.6)	33 (39.7)	12 (19.9)	2 (2.0)	0 (0.2)
Carlsberg	75 (99.3)	61 (79.4)	45 (59.6)	32 (39.7)	20 (19.9)	0 (2.0)	0 (0.2)
DS 1912	94 (98.3)	66 (78.6)	48 (59.0)	26 (39.3)	11 (19.6)	2 (2.0)	1 (0.2)
ISS	144 (98.2)	117 (78.5)	87 (58.9)	57 (39.3)	22 (19.6)	5 (2.0)	0 (0.2)
Novo B	93 (99.3)	75 (79.4)	52 (59.6)	36 (39.7)	18 (19.9)	2 (2.0)	0 (0.2)
Svendborg	102 (99.3)	87 (79.4)	60 (59.6)	34 (39.7)	14 (19.9)	3 (2.0)	1 (0.2)

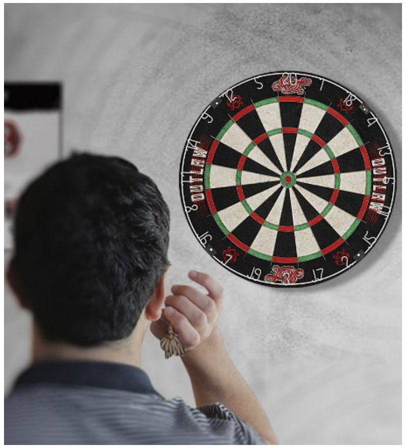
GARCH based extreme value method, unconditional

	95%	96%	97%	98%	99%	99.9%	99.99%
Portfolio	107 (98.1)	90 (78.4)	66 (58.8)	44 (39.2)	16 (19.6)	0 (2.0)	0 (0.2)
S&P500	98 (143.8)	77 (115.1)	55 (86.3)	30 (57.5)	14 (28.8.0)	2 (2.9)	0 (0.3)
B&O	82 (99.3)	65 (79.4)	52 (59.6)	34 (39.7)	14 (19.9)	0 (2.0)	0 (0.2)
Carlsberg	88 (98.8)	77 (79.0)	62 (59.3)	37 (39.5)	22 (19.8)	0 (2.0)	0 (0.2)
DS 1912	108 (94.8)	87 (75.8)	69 (56.9)	43 (37.9)	20 (19.0)	1 (2.0)	0 (0.2)
ISS	114 (81.7)	100 (65.3)	70 (49.0)	41 (32.7)	16 (16.3)	1 (1.6)	0 (0.2)
Novo B	106 (99.2)	87 (79.4)	62 (59.5)	49 (39.7)	25 (19.8)	2 (2.0)	0 (0.2)
Svendborg	115 (98.6)	100 (78.8)	76 (59.1)	56 (39.4)	27 (19.7)	2 (2.0)	0 (0.2)

- compute VaR from the first six years of data, see if it "is violated", i.e. if next days return is lower than VaR, repeat again using six years of data but starting one day later, two days later, … count number of violations

- expected no. of violations in parentheses

A final important reminder



For a stationary sequence block maxima (of course) are stochastically larger than one-day values, and block maxima over longer blocks are stochastically larger than maxima over shorter blocks.

Why is this picture here? Try to understand why.

(That a random variable is stochastically larger than another one means that it's distribution function lies to the right of the other one. Think carefully through what this means practically and why it is true)

Always make plots of your data/time series. This makes it possible to see if your results and choices are OK/good