

Examples on stability of equilibrium points

Example 1. Simple strong Lyapunov function.

Exercise 15 Show that $(x(t), y(t)) = (0, 0)$ is an asymptotically stable solution of

$$\begin{cases} \dot{x} = -x^3 + 2y^3 \\ \dot{y} = -2xy^2. \end{cases}$$

5

Example 2. Stability by Linearization

For the following system of equations find all equilibrium points and investigate their stability and their type by linearization.

$$\begin{cases} x' = \ln(2 - y^2) \\ y' = \exp(x) - \exp(y) \end{cases}$$

1. **Solution.** There are two equilibrium points: $x_1 = (1, 1)$ and $x_2 = (-1, -1)$.

The Jacobian of the right hand side is: $\begin{bmatrix} 0 & -2\frac{y}{-y^2+2} \\ e^x & -e^y \end{bmatrix}$. Its values in x_1 and x_2 are $A_1 = \begin{bmatrix} 0 & -2 \\ e & -e \end{bmatrix}$, and $A_2 = \begin{bmatrix} 0 & 2 \\ 1/e & -1/e \end{bmatrix}$. The eigenvalues to A_1 are $-\frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 8e}$, and $\frac{1}{2}\sqrt{e^2 - 8e} - \frac{1}{2}e$ that are conjugate complex numbers with negative real parts. Therefore we observe stable spiral around the equilibrium point x_1 . The eigenvalues to A_2 are , eigenvalues: $\frac{1}{e}(-\frac{1}{2}\sqrt{8e+1} - \frac{1}{2})$, $\frac{1}{e}(\frac{1}{2}\sqrt{8e+1} - \frac{1}{2})$, one positive and one negative. Therefore x_2 is a saddle point and is unstable.

Example 3.

Consider the following system of ODEs:
$$\begin{cases} x' = 2y \\ y' = -x - (1 - x^2)y \end{cases}.$$

Show the asymptotic stability of the equilibrium point in the origin and find its domain of attraction.

Solution.

We try the test function $V(x, y) = x^2 + Ay^2$ that leads to cancellation of mixed terms in the directional derivative V_f along trajectories:

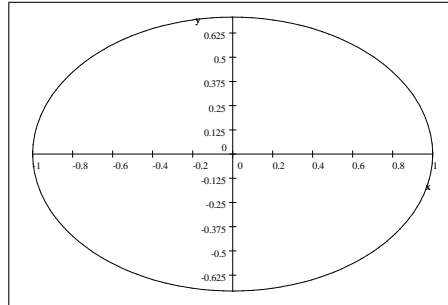
$$V_f(x, y) = \nabla V \cdot f(x) = 2x2y + (2Ay(-x - (1 - x^2)y)) = 4xy - 2Axy - 2Ay^2(1 - x^2)$$

Choose $A = 2$ to cancel indefinite terms. $V(x, y) = x^2 + 2y^2$

$V_f(x, y) = 4xy - 4xy - 4y^2(1 - x^2) = -4y^2(1 - x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point.

Checking the behavior of the system on the set of zeroes to $V_f(x, y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x, y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. The LaSalle invariance principle implies that the origin is asymptotically stable.

The domain of attraction is the largest set bounded by a level set of $V(x, y) = x^2 + 2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that it touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$ and the boundary of the region (domain) of attraction is the ellipse $x^2 + 2y^2 = 1$ with halves of axes 1 and $\sqrt{0.5}$:



How to find a Lyapunov function?

If the right hand side of the equation is a higher degree polynomial, then it is often convenient to find a Lyapunov's function in a systematic way in the form of polynomial with unknown coefficients and unknown even degrees like $2m$.

Consider the system

$$\begin{aligned}x' &= -3x^3 - y \\y' &= x^5 - 2y^3\end{aligned}$$

Try a test function $V(x, y) = ax^{2m} + by^{2n}$, $a, b > 0$.

$$\begin{aligned}V_f(x, y) &= \nabla V \cdot f(x, y) = \\&= a2m(x)^{2m-1} \cdot (-3x^3 - y) + b2n(y)^{2n-1} (x^5 - 2y^3) \\&= \underbrace{-6amx^{2m+2}}_{good < 0} - \underbrace{2ma(x)^{2m-1}y}_{bad-indefinite} + \underbrace{2nby^{2n-1}x^5}_{bad-indefinite} - \underbrace{4nby^{2n+2}}_{good < 0}\end{aligned}$$

We choose first powers m and n so that indefinit terms would have same powers of x and y .

$$\begin{aligned}2m - 1 &= 5; \implies m = 3 \\2n - 1 &= 1; \implies n = 1\end{aligned}$$

Then $V_f(x, y) = -18ax^8 - 6x^5y + 2bx^5y - 4nby^4$. We choose $a = 1$ and $b = 3$ to cancel indefinite terms. Then

$$\begin{aligned}V(x, y) &= x^6 + 3y^2 \\V_f(x, y) &= -18x^8 - 12y^4 < 0, \quad (x, y) \neq (0, 0)\end{aligned}$$

Therefore V is a strong Lyapunov's function in the whole plane and the equilibrium is a globally asymptotically stable equilibrium point, because $V(x, y) = x^6 + 3y^2 \rightarrow \infty$ as $\|(x, y)\| \rightarrow \infty$.

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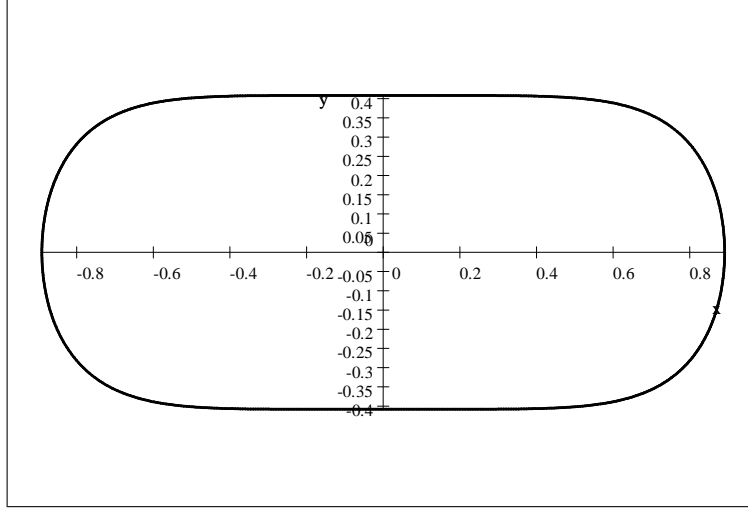
Example 4. Investigate stability of the equilibrium point in the origin for the following system:

$$\begin{aligned}x' &= -y - x^3 \\y' &= x^5\end{aligned}$$

We try our simplest choice of the Lyapunov function: $V(x, y) = x^2 + y^2$ and arrive to

$$V_f(x, y) = -2xy - 2x^4 + 2yx^5$$

It does not work because the expression $V_f(x, y)$ includes two indefinite terms: $2xy$ and $2yx^5$ that change sign around the origin. We try a more flexible expression by looking on particular expressions in the right hand side of the equation: $V(x, y) = x^6 + \alpha y^2$ where $\partial V / \partial x = 6x^5$ with the same power of x as in the equation, and the parameter α that can be adjusted later. V is a positive definite function: $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$. The level sets to V look as flattened in y - direction ellipses. The curve $x^6 + 3y^2 = 0.5$ is depicted:



$$V_f(x, y) = 6x^5(-y - x^3) + 2\alpha yx^5 = -6x^5y + 2\alpha x^5y - 6x^8$$

We get again two indefinite terms, but they are proportional and the choice $\alpha = 3$ cancels them:

$$V_f(x, y) = -6x^8 \leq 0$$

Therefore the origin is a stable equilibrium point. $V_f(x, y) = 0$ on the whole y -axis that in our "general" theory is denoted by $V_f^{-1}(0)$. We check invariant sets of the system on the set $V_f^{-1}(0)$. We observe that $x' = -x^3$ (only this fact is important) and $y' = 0$ (it does not matter for $V_f^{-1}(0)$ that is y -axis). Therefore $\{0\}$ is the only invariant set on the y - axis. Trajectories starting on the y - axis go across it in all points except $\{0\}$. The LaSalle's invariance principle implies that all trajectories approach $\{0\}$ as t tends to infinity and the origin is asymptotically stable.

The test function $V(z) \rightarrow \infty$ as $\|z\| \rightarrow \infty$. It implies that the whole plain is a region or domain of attraction for the equilibrium point in the origin.

A very useful inequality in analysis is

Young's inequality

Lemma. If $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for every pair of numbers $p, q \in (1, \infty)$ satisfying the conjugacy relation.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The simplest example of Young's inequality:

$$ab \leq \frac{1}{2} (a^2 + b^2)$$

Example 5.

Consider the Lienard equation: $x'' + x' + g(x) = 0$, and investigate stability of the equilibrium in the origin. The second order equation can be rewritten as a system $z' = f(z)$:

$$\begin{aligned} x' &= y \\ y' &= -g(x) - y \end{aligned}$$

where g satisfies the following hypothesis: g is continuously differentialble for $|x| < k$ for some $k > 0$, $xg(x) > 0$, $x \neq 0$.

Solution.

Physically this equation is a Newton equation for a non-linear spring. For example if $g(x) = \sin(x)$ it describes a pendulum with friction where air resistance is proportional to velocity.

A Lyapunov function is naturally to choose as a total energy of the system:

$$V(x, y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$$

Indeed it is positive definite in the region $\Omega = \{(x, y) : |x| < k\}$ because $g(s)s > 0$ in Ω according to given conditions. The directional derivative of V along f is

$$V_f(x, y) = y(-g(x) - y) + g(x)y = -(y)^2$$

V is a Lyapunov's function, but not strong because $V_f(x, y) = 0$ on the whole x - axis. Therefore $V_f^{-1}(0)$ is the whole x - axis. Checking values of f on $V_f^{-1}(0)$ we observe that trajectories of the system are orthogonal to $V_f^{-1}(0)$ in all points on $V_f^{-1}(0)$ except the origin. It implies that $\{0\}$ is the only invariant set on $V_f^{-1}(0)$ that attracts all trajectories starting in a small neighborhood of the origin. Therefore the origin is asymptotically stable.

Our next problem is to find a possibly large domain or region of attraction for the equilibrium point. If we find a closed level set for V in Ω , it will be a boundary for a domain of attraction. It might not be the largest possible and depends on a clever choice of Lyapunov's function V .

We cannot solve this problem for a general expression $V(x, y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$.

Example 6.

Find all equilibriums, investigate their stability properties and find possible regions of attraction.

Choose a particular $g(x) = x + x^2$ in the previous example.

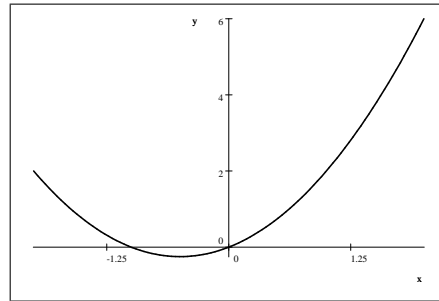
$$\begin{aligned}x' &= y \\ y' &= -(x + x^2) - y\end{aligned}$$

Observe that the system has two equilibrium points: $(-1, 0)$ and $(0, 0)$

Linearization gives Jacoby matrix $A(x, y) = \begin{bmatrix} 0 & 1 \\ -1 - 2x & -1 \end{bmatrix}$; $A(-1, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ Observe that $\det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 0 - 1 = -1 < 0$ it implies by the Grobman - Hartman theorem, that $(-1, 0)$ is a saddle point.

$A(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $\det \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = 1 > 0$, $trace \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -1 < 0$,

$(trace A(0, 0))^2 / 4 = 1/4 < 1 = \det A(0, 0)$. It implies that the origin is an asymptotically stable focus for the linearized system and is asymptotically stable for the original system.



$$g(x) = x + x^2$$

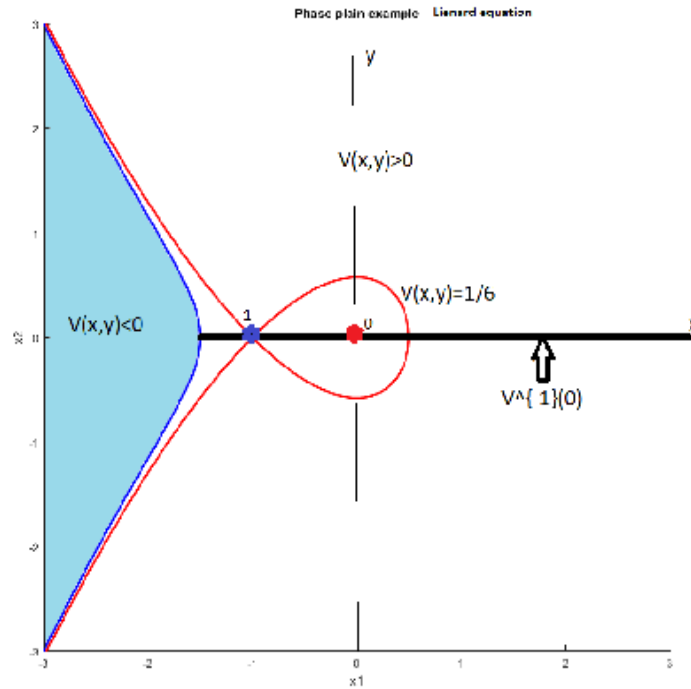
We can find an explicit expression for the Lyapunov's function $V(x, y) = \frac{(y)^2}{2} + \int_0^x g(s)ds$.

$$V(x, y) = \frac{(x)^2}{2} + \frac{(x)^3}{3} + \frac{(y)^2}{2}$$

This function is positive definite on the set $\Omega = \{(y)^2 > -(x)^2 - \frac{2}{3}(x)^3\}$

The level set $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{1}{6}$ is depicted by the red line.

The level set $\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 = 0$ is depicted by the blue line. We will investigate them analytically a bit later.



$V_f(x, y) = \nabla V(x, y) \cdot f = xy + (x)^2 y - (y)^2 - xy - (x)^2 y = -(y)^2 \leq 0$ valid in the whole plane \mathbb{R}^2 .

We check which invariant sets are contained in $V_f^{-1}(0)$ on Ω that is a part of x - axis $\{(x, 0) : x > -3/2\}$ that is a thick black line on the picture above.

Notice that $V_f^{-1}(0)$ on Ω contains two equilibrium points $(-1, 0)$ and $(0, 0)$ and they both are invariant sets. We like to find a largest domain $\Omega_1 \subset \Omega$ bounded by a part of a level set of V such that Ω_1 does not include

the point $(-1, 0)$. Then Ω_1 contains only one invariant set that is the origin $(0, 0)$. This set Ω_1 is the domain of attraction for the asymptotically stable equilibrium in $(0, 0)$.

Such largest level set of V must go through the second equilibrium point $(-1, 0)$ and it's value there is $V(x, y) = V(-1, 0) = 1/6$. The domain of attraction Ω^* is the egg - shaped domain bounded by the closed curve $(y)^2 = 1/3 - ((x)^2 + \frac{2}{3}(x)^3)$ or as a union of two explicit branches:

$$y = \pm \sqrt{1/3 - \left((x)^2 + \frac{2}{3}(x)^3 \right)}$$

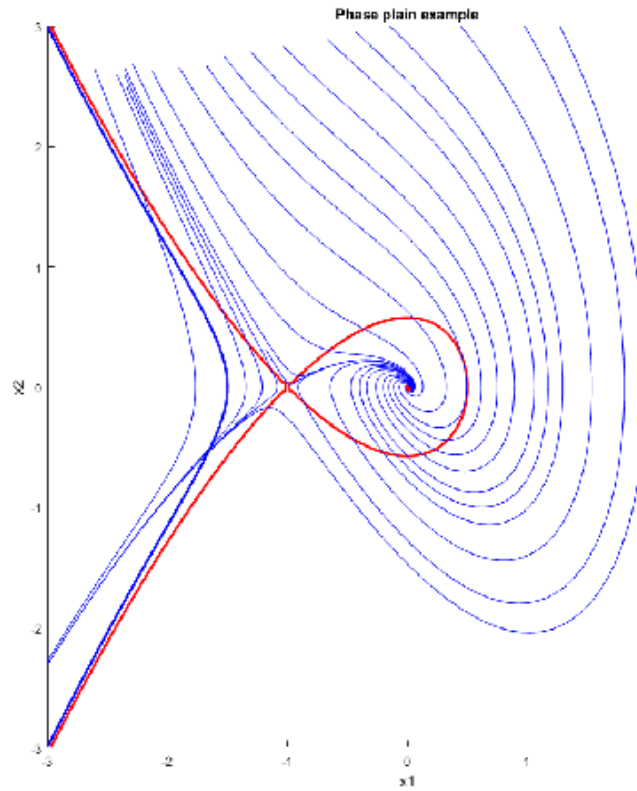
It is a part of the red level set on the picture. To see that this curve is closed we consider derivative of the function

$\frac{d}{dx} \left(1/3 - ((x)^2 + \frac{2}{3}(x)^3) \right) = -2x - 2x^2 = (-2)x(x+1)$. It implies that the functions has a maximum in $x = 0$, and minimum at $x = -1$. $V(x)$ has zero in $x = -1$ and another zero in $x = 1/2$:

$$1/3 - ((x)^2 + \frac{2}{3}(x)^3) \Big|_{x=1/2} = 1/3 - ((1/2)^2 + \frac{2}{3}(1/2)^3) = 1/3 - ((1/4) + \frac{1}{3}(1/4)) = 1/3 - 1/3 = 0,$$

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One can try to find an even larger region of attraction Ω^{**} for the equilibrium point in the origin. It cannot include the equilibrium in $(-1, 0)$ because it is unstable (a saddle point). We can extend Ω_1 to a rectangle $[-1, 0] \times [0, \sqrt{3}/3]$ in the second quadrant by checking signs of x' and y' on it's left and upper sides. Actual region of attraction is even a bit larger as one can see on the phase portrait



Example 7. Exercise 5.13 from L.R.

Investigate stability of the equilibrium point in the origin and find a possible domain of attraction for the following system.

$$\begin{aligned}x_1' &= -x_2(1 + x_1x_2) \\x_2' &= 2x_1\end{aligned}$$

We try choose the Lyapunov function V as

$$V(x_1, x_2) = 2x_1^2 + x_2^2$$

We could try first a function $V(x_1, x_2) = ax_1^2 + x_2^2$, check V_f and then

decide which value a suites best.

$$\begin{aligned} V_f(x_1, x_2) &= \nabla V \cdot f(x_1, x_2) = -2ax_1x_2(1 + x_1x_2) + 2x_22x_1 \\ &= 4x_1x_2 - 2ax_1x_2 - 2ax_1^2x_2^2 = -2ax_1^2x_2^2 \leq 0 \\ \text{for } a &= 2 \end{aligned}$$

We conclude that the equilibriom 0 is stable. $V_f(x_1, x_2) = -2ax_1^2x_2^2 = 0$ on both coordinate axes. We check which invariant sets are contained in $V_f^{-1}(0)$.

If $x_1 = 0$, then $x'_1 = -x_2$, $x'_2 = 0$. Therefore only $\{0\}$ is an invariant set on the x_2 axis.

If $x_2 = 0$, then $x'_1 = 0$, $x'_2 = 2x_1$. Therefore only $\{0\}$ is an invariant set on the x_1 axis.

Trajectories $\varphi(t, \xi)$ starting inside ellipses $V(x_1, x_2) = 2x_1^2 + x_2^2 = C > 0$ are contained inside these ellipses because $\nabla V \cdot f(x) \leq 0$. It implies that their positive orbits $O_+(\xi)$ are bounded and have compact closure in \mathbb{R}^2 .

It implies according to the LaSalle's theorem that all these solutions $\varphi(t, \xi)$ approach the maximal invariant set in $V_f^{-1}(0)$ that in our particular case consists of just one point $(0, 0)$. Therefore the equilibrium point in the origin is asymptotically stable. It is also globally stable because the Lyapunov function $V(x)$ tends to infinity as $\|x\| \rightarrow \infty$, making that arbitrary large elliptic discs from the family $2x_1^2 + x_2^2 < C$ are regions of attraction.

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Example 8. This example demonstrates how to use Cauchy and Young inequalities for estimating $V_f(x, y)$

Two following inequalities are convenient to estimate absolute value of "bad" indefinite terms like xy , x^3y^2 , xy^2 , et.c.

Cauchy inequality:

$$|x| |y| \leq \frac{1}{2} (x^2 + y^2)$$

Young inequality If $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for every pair of numbers $p, q \in (1, \infty)$ satisfying the conjugacy relation.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The Cauchy inequality is the simplest example of Young's inequality for $p = q = 2$.

Consider the following system of ODE:
$$\begin{cases} x' = -x - 2y + xy^2 \\ y' = 3x - 3y + y^3 \end{cases}$$

Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

Hint: applying Lyapunov's theorem, use the Cauchy inequality $2xy \leq (x^2 + y^2)$ to estimate indefinite terms with xy . (4p)

Solution. Choose a test function $V(x, y) = \frac{1}{2} (x^2 + y^2)$

$$\begin{aligned} V_f &= x(-x - 2y + xy^2) + y(3x - 3y + y^3) = xy - x^2 - 3y^2 + y^4 + x^2y^2 \\ &= -x^2(1 - y^2) - y^2(3 - y^2) + xy \leq 0 \quad \text{????} \end{aligned}$$

We apply the Cauchy inequality $|x| |y| \leq \frac{1}{2} (x^2 + y^2)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

$$V_f \leq -x^2(0.5 - y^2) - y^2(2.5 - y^2)$$

It implies that $V_f(x, y) < 0$ for $(x, y) \neq (0, 0)$ and $|y| < 1/\sqrt{2}$. Therefore the Lyapunov function is strong and the origin is asymptotically stable.

The attracting region is bounded by the largest level set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely $(x^2 + y^2) < 1/2$.

Another more clever choice of a test function is $V(x, y) = 3x^2 + 2y^2$.

$$V_f = 6x(-x - 2y + xy^2) + 4y(3x - 3y + y^3) = 4y^4 - 12y^2 - 6x^2 + 6x^2y^2 = -4y^2(3 - y^2) - 6x^2(1 - y^2) < 0$$

for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ is a domain of attraction for the asymptotically stable equilibrium in the origin.

One can also observe the asymptotic stability of the origin by linearization with variational matrix

$$A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}, \text{ with characteristic polynomial: } \lambda^2 + 4\lambda + 9 = 0,$$

and calculating eigenvalues: $-i\sqrt{5} - 2, i\sqrt{5} - 2$ with $\operatorname{Re} \lambda < 0$. But the linearization gives no information about the region of attraction. ■

Example 9 (on instability)

Consider the following system of ODEs. Prove the instability of the equilibrium point in the origin, of the following system

$$\begin{cases} x' = x^5 + y^3 \\ y' = x^3 - y^5 \end{cases} \quad (4p)$$

using the test function $V(x, y) = x^4 - y^4$ and Lyapunov's instability theorem.

Solution.

Denoting $f(x, y) = \begin{bmatrix} x^5 + y^3 \\ x^3 - y^5 \end{bmatrix}$, consider how the function $V(x, y) = x^4 - y^4$ changes along trajectories of the system.

$$V_x(x, y) = f(x, y) \cdot \nabla V(x, y) = \begin{bmatrix} x^5 + y^3 \\ x^3 - y^5 \end{bmatrix} \cdot \begin{bmatrix} 4x^3 \\ -4y^3 \end{bmatrix} = x^5 4x^3 + y^3 4x^3 - x^3 4y^3 + y^5 4y^3 = x^5 4x^3 + y^5 4y^3 = 4(x^8 + y^8) > 0.$$

Point out that the function $V(x, y) = x^4 - y^4$ is positive in the cone $|y| < |x|$ or simply along the line $y = x/2$, arbitrarily close to the origin. It implies according to the instability theorem, that the origin is an unstable equilibrium. ■

How to find a strong Lyapunov's function?

Example 10.

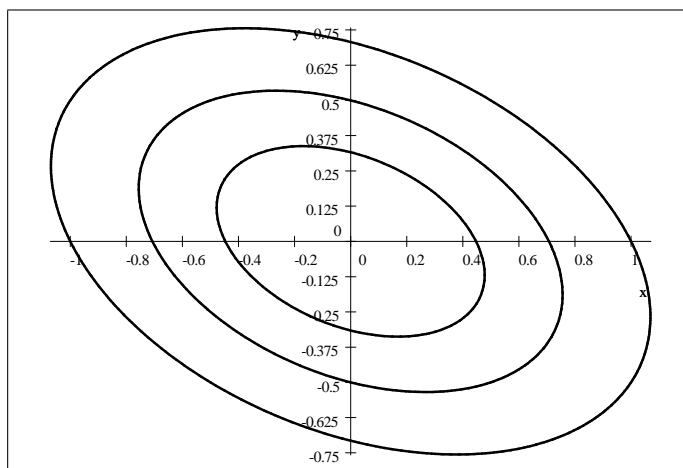
It is theoretically possible to find a strong Lyapunov function for the same system as in the Example 3.

Looking on the previous week Lyapunovs function $x^6 + 3y^2$ we see that it's "weakness" followed from the fact that both level sets of V and velocities of the system were orthogonal to the y - axis. It implied that $V_f(z) = 0$ on the y - axis. To go around this problem a strong Lyapunov function must have level sets that deviate slightly from the normal to the y - axis. Adding a relatively small indefinite term xy^3 to the function $x^6 + 3y^2$ we get this effect. A level set corresponding $x^6 + xy^3 + 3y^2 = 0.7$ of this new Lyapunovs function looks as a slightly rotated version of level sets for the previous (weak) Lyapunovs function.

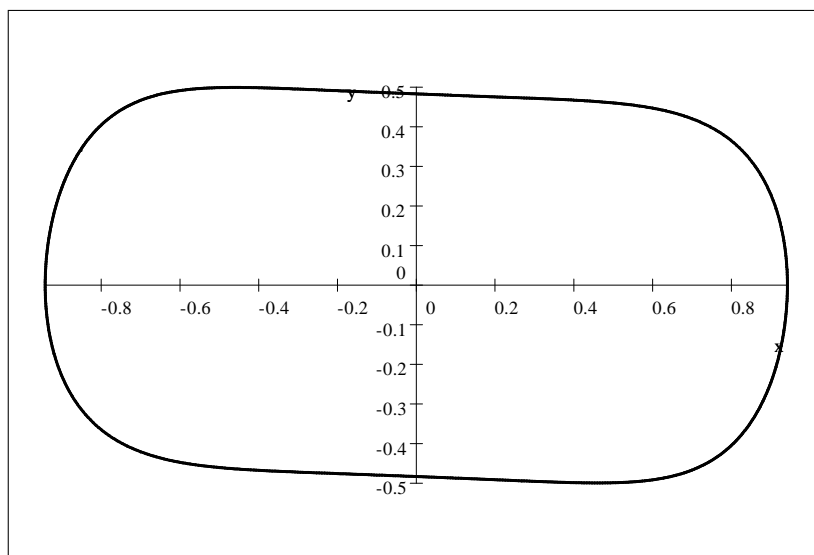
Why like that ? Take a simpler example with an ellipse curve $x^2 + 2y^2 = 1$ and another that is $x^2 + xy + 2y^2 = 1$

This quadratic form is positive definite: the matrix is $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$. A quadratic form $\mathbf{x}^T A \mathbf{x} = Q(\mathbf{x})$ with $\mathbf{x} = [x, y]^T$ is positive definite if and only if $\det A > 0$ and all submatrices A_i from the upper left corner have positive determinants: $\det A_i > 0$.

Level sets of the positive definite quadratic form with mixed tems like $x^2 + xy + 2y^2$ are ellipses with symmetry axes (that are orthogonal eigenvectors to A) and are rotated with respect to coordinate axes:



We try to introduce the test function $V(x, y) = x^6 + xy^3 + 3y^2$ with an indefinite mixed term xy^3 added, that would similarly with the ellipses, give slightly rotated level sets so that trajectories would cross them strictly inside on the y - axis:



We claim that the test function $V(x, y) = x^6 + xy^3 + 3y^2$ is positive definite and is a strong Lyapunovs function namely that $V_f(x, y) < 0$ for $(x, y) \neq (0, 0)$.

Because of the geometry of the vector field f of our equation $z' = f(z)$ velocities on the y axis cross such level sets strictly towards inside, implying the desired strict inequality $V_f(z) < 0$, $z \neq 0$ on the y axis. We need to check that $V(x, y) = x^6 + xy^3 + 3y^2$ is positive definite (it is not trivial) and to show that $V_f(z) < 0$, $z \neq 0$ for all $z \in \mathbb{R}^2$ (it requires some non-trivial analysis).