# 1

## Introduction to the modelling poject and to nullclines

**Logistic equation and two species competition model.**

Let $x_i(t), i = 1, 2.$ be populations of two species. Each of the species grows with intrinsic growth rate $r_i$ in case when infinite resources are available: $x_i' = r_i x_i$, $r_i > 0$.

Limited resources lead to competition within the population and a limited growth rate for the large size of the population: $r_i(1 - \frac{x_i}{K_i})$. This model is called the logistic equation:

$$x_i' = r_i x_i \left(1 - \frac{x_i}{K_i}\right) \tag{1}$$

The competition between different species leads to a decrease in each population with the decreasing rate proportional to the competitor population size: $-\alpha_1 x_2$ for the population $x_1$ and $-\alpha_2 x_1$ for the population $x_2$ with competition coefficients $\alpha_1 > 0$ and $\alpha_2 > 0$. The corresponding system of equations describes the evolution of two competing species:

$$\begin{aligned} x_1' &= r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\ x_2' &= r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_2 x_1 \end{aligned} \tag{2}$$

**What are natural questions to pose about an environmental model?**

1. Existence and uniqueness of solutions

2. Positivity of solutions is important for chemical and environmetal models where variables have meaning of mass or number of individuals.

3. We look for sustainable evolutions like equilibrium points, in particular stable and asymptotically stable equilibriums (states of coexistence), periodic solutions.

4. Find if all solutions are bounded when $t \to \infty$ ?

5. Phase portrait gives a feeling of the global picture of all possible solutions for different initial states. We would like to be able to classify qualitatively different pictures of the phase portrait depending on combinations of parameters in the system: large or small competition coeffitients $\alpha_1 > 0$ and $\alpha_2 > 0$ in comparison with $r_1, K_1$ et.c.

6. Nullclines are lines where one of the components of velocity is zero. They separate areas where components of velocity are positie and negative.

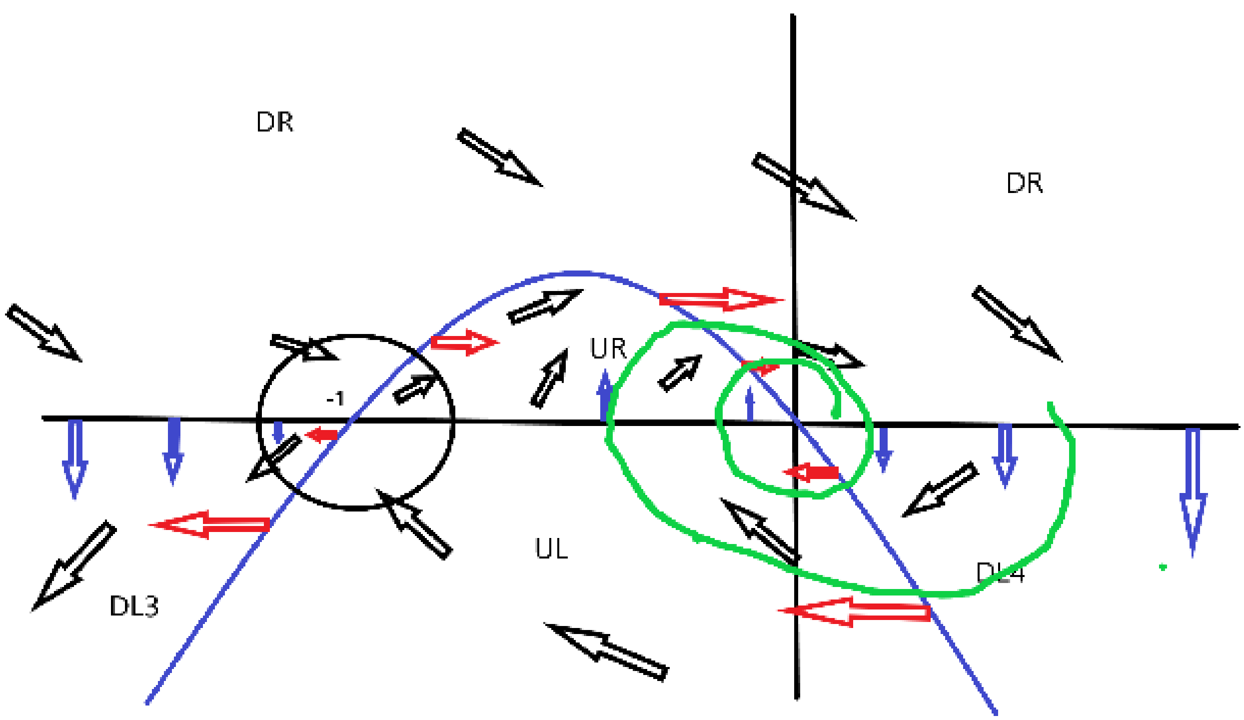
**Example on the application of nullclines**

$$\begin{cases} x' = & y \\ y' = & -y - x - x^2 \end{cases}$$

Nullclines of this system have equations $y = 0$ ( $x$ - nullcline) that is the $x$ - axis, and $y = -x - x^2$, that is a parabola throught points $(-1, 0)$ and $(0, 0)$ ($y$ - nullcline). See the picture.

Equilibrium points are intersection points of different nullclines: $(0, 0)$ and $(-1, 0)$.

Velocities on the nullclines have only one component non-zero: $x$ - component on the $y$- nullcline and $y$ - component on the $x$ - nullcline. Velocity vectors on nullclines are illustrated as red and blue vectors on the sketch below.




We observe that the plane is divided by nullclines into five domains where components of velocity have constant sign.

We mark these domains by abbreviations DR for down-right, DL3 for down - left in the third quadrant, UR - for up-right, UL - for up - left and DL4 - for for the down left in the fourth quadrant.

We sketch directions of velocities on the nullclines and taking into account directions of velocities can make conclusions about how trajectories of solutions to the differential equation can enter and leave these domains through different parts of nullclines. It will give us a rather rich picture of the phase protrait and can be an argument for stating instability of one of the equilibrium points.

The domain **DR** is bounded by two intervals on the $x$-axis: $(-\infty, -1)$ and $(0, \infty)$ that are parts of the $x$ - nullcline, and by the part of the parabola $y = -x - x^2$ with $x \in (-1, 0)$ that is part of $y$ - nullcline. Trajectories leave this domain in the direction straight down through the intervals $(-\infty, -1)$ and $(0, \infty)$ and leave it in the direction straight to the right through the part of the parabola $y = -x - x^2$ with $x \in (-1, -0.5)$.Trajectories enter this domain in the direction straight to the right through the right through the part of the parabola $y = -x - x^2$ with $x \in (-0.5, 0)$.

The domain **DL4** is bounded by the interval $(0, \infty)$ on the $x$ - axis that is a part of the $x$ - nullcline and by the part of the parabola $y = -x - x^2$ with $x \in (0, \infty)$ that is part of $y$ - nullcline. Trajectories enter this domain through the first line straight down and leave it through the second line straight to the left.

The domain **UR** is bounded by the part of the parabola $y = -x - x^2$ with $x \in (-1, 0)$ that is part of $y$ - nullcline and by the interval $(-1, 0)$ on the $x$ - axis that is part of $x$ - nullcline.

Trajectories enter this domain through the part of the parabola $y = -x - x^2$ with $x \in (-1, 0.5)$ straight to the right and throught the interval $(-1, 0)$ on the $x$ - axis in the direction straight up. Trajectories leave this domain throught the part of the parabola $y = -x - x^2$ with $x \in (-0.5, 0)$ in the direction straight to the right.

The domain **UL** is bounded by the interval $(-1, 0)$ on the $x$ - axis that is part of $x$ - nullcline and by two branches of the parabola $y = -x - x^2$ with $x \in (0, \infty)$ and $x \in (-\infty, -1)$.

Trajectories enter this domain through the part of the parabola $y = -x - x^2$ with $x \in (0, \infty)$ in the direction straight to the left. Trajectories leave this domain throught the part of the parabola $y = -x - x^2$ with $x \in (-\infty, -1)$ in the direction straight to the left. Trajectories also leave this domain throught the interval $(-1, 0)$ on the $x$ - axis in the direction straight up.

The domain DL3 is bounded by the interval $(-\infty, -1)$ on the $x$ - axis that is a part of $x$ - nullcline and by the branch of the parabola $y = -x - x^2$ with $x \in (-\infty, -1)$.Trajectories enter this domain throught the part of the parabola $y = -x - x^2$ with $x \in (-\infty, -1)$ in the direction straight to the left. Trajectories enter this domain in the direction straight down through the interval $(-\infty, -1)$.

These observations can make qualitative conclusions about behavior of trajectories in small vicinities of equilibrium points $(-1, 0)$ and $(0, 0)$.

We observe looking on directions of trajectories in domains UR, DR, DL and UL that trajectories must go in spirals around the equilibrium point $(0, 0)$. But we can not make the conclusion about stability or instability of the equilibrium in $(0, 0)$.

We observe that any trajectory $x(t)$ starting at a point $\xi$ arbitrarily close to the equilibrium point $(-1, 0)$ inside domains UR or DL move out of the circle of radius $r = 0.25$ (or smaller) still staying in the same domain. It happens so because such trajectory has both components of velocity strictly positive (in UR) or strictly negative (in DL) before it leaves the circle. It implies that this equilibrium point is unstable. We can sketch phase portrait around this equilibrium and guess that it might be a saddle point.

## 1.1 Stability of equilibrium points by linearization.

We consider in this chapter of the course properties of solutions of I.V.P to nonlinear autonomous systems of ODEs

$$x'(t) = f(x(t)), \quad x(0) = \xi \tag{3}$$

where $f : G \to \mathbb{R}^N$ is locally Lipschitz with respect to $x$. $J$ is and interval and $G \subset \mathbb{R}^N$ is a non-empty open set.

We will consider in this chapter of the course the stability of equilibrium points $x_*$ of such nonlinear systems ($f(x_*) = 0$) in connection with properties of corresponding linearized systems in the form

$$y'(t) = Ay(t) \tag{4}$$

where $A = \frac{Df}{Dx}(x_*)$ is a Jacoby matrix of the function $f$ calculated in an equilibrium point of interest.

**Definition.** (p. 115, L.R.) A function $f$ is called locally Lipschitz in $G$ if for any point $y \in G$ there is a neighborhood $V(y)$ and a number $L > 0$ (depending on $V(y)$) such that for any $v, w \in V(y)$

$$\|f(v) - f(w)\| \leq L \|v - w\|$$

**Example.** Functions having continuous partial derivatives are locally Lipschitz function. (Exercise)

**Definition**. A solution $x(t) : I \to \mathbb{R}^N$ is called **maximal solution** to an I.V.P. if it cannot be extended to a larger time interval.

## 1.2 Peano existence theorem.

The theorem by Peano, states that if $f : G \to \mathbb{R}^N$ is continuous, the the I.V.P. (3) above has a solution (not unique!!!) for any $\xi \in G$ on some, might be small time interval $(-\delta, \delta)$. (Theorems 4.2, p. 102; )

We will consider Peano theorem it at the end of the course.

## 1.3 Picard and Lindelöf's existence and uniqueness theorem.

The theorem by Picard and Lindelöf, states that if $f : G \to \mathbb{R}^N$ is locally Lipschitz, then the I.V.P. (3) above has a unique solution for any $\xi \in G$ on some, might be small time interval $(-\delta, \delta)$. (Theorems 4.17, p. 118; Theorem 4.22, p.122.)

We will formulate it in a more general form and will prove it at the end of the course.

## 1.4 Definition of stable equilibrium points (repetition).

**Definition**. A point $x_* \in G$ is called an equilibrium point to the equation (3) if $f(x_*) = 0$.

The corresponding solution $x(t) \equiv x_*$ is called an equilibrium solution.

**Definition.** (5.1, p. 169, L.R.)

The equilibrium point $x_*$ is said to be stable if, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any maximal solution $x : I \to G$ to (3) such that $0 \in I$ and $\|x(0) - x_*\| \leq \delta$ we have $\|x(t) - x_*\| \leq \varepsilon$ for any $t \in I \cap \mathbb{R}_+$. Below a picture is given in the case $x_* = 0$.

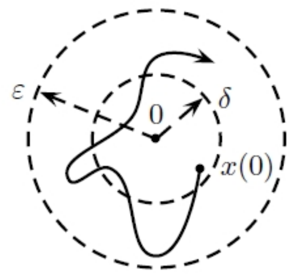



**Figure 5.1** Stable equilibrium

**Definition.** (5.14, p. 182, L.R.)

The equilibrium point $x_*$ of (3) is said to be *attractive* if there is $\delta > 0$ such that for every $\xi \in G$ with $\|\xi - x_*\| \leq \delta$ the following properties hold: the solution $x(t) = \varphi(t, \xi)$ to I.V.P. with $x(0) = \xi$ exists on $\mathbb{R}_+$and $\varphi(t, \xi) \to x_*$ as $t \to \infty$.

**Definition.** We say that the equilibrium $x_*$ is **asymptotically stable** if it is both stable and attractive.

In the analysis of stability we will always choose a system of coordinates so that the origin coincides with the equilibrium point. In the course book this agreement is applied even in the definition of stability.

**Definition.** The equilibrium point $x_*$ is said to be *unstable* if it is not stable. It means that there is a $\varepsilon_0 > 0$, such that for any $\delta > 0$ there is point $x(0) : \|x(0) - x_*\| \leq \delta$ such that for some $t_0 \in I$ we have $\|x(t_0) - x_*\| > \varepsilon_0$.(a formal negation to the definition of stability)

## 1.5 Stability and instability of the equilibrium point in the origin for autonomous linear systems.

Origin is an equilibrium point for all linear systems of ODE. If the matrix $A$ is degenerate namely if $\det(A) = 0$, there can appear lines or in higher dimensions - hyperplanes of equilibrium points in addition to the origin, corresponding to the non-trivial kernel of the matrix $A$.

The following general statement about stability and instability of the equilibrium in the origin for arbitrary autonomous linear systems of ODEs follow immediately from the Corollary 2.13 in L.&R.

**Theorem.** (Propositions 5.23, 5.24, 5.25, pp. 189-190, L.R.)

Let $A \in \mathbb{C}^{N\times N}$ be a complex matrix.

Then three following statements are valid for the system $x'(t) = Ax(t)$

1. The origin is asymptotically stable equilibrium point if and only if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

2. The equilibrium point in the origin is stable if and only if $\operatorname{Re}\lambda \leq 0$ for all $\lambda \in \sigma(A)$ and all eigenvalues $\lambda$ with $\operatorname{Re}\lambda = 0$ are semisimple (the number of linearly independent eigenvectors to $\lambda$ is equal to the algebraic multiplicity $m(\lambda)$ of $\lambda$)

3. The equilibrium point in the origin is unstable if and only if there is at least one eigenvalue $\lambda$ with $\operatorname{Re}\lambda > 0$ or an eigenvalue $\lambda$ with $\operatorname{Re}\lambda = 0$ that is not semisimple.

   (**3.** is a direct consequence of **2.** )

   Proof of this theorem is a simple exrecise based on the definitions of stability and asymptotic stability and on the Corollary 2.13 about the properties of $\|\exp(At)\|$ and $\|\exp(At)\xi\|$ .

**Definition.** Matrix $A$ with the property $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$ is called **Hurwitz matrix.**

**OBS!**

We point out that the origin is an asymptotically stable equilibrtium for systems of ODEs $x'(t) = Ax(t)$ with Hurwitz matrix $A$.

One of the goals of the lecture is to show that under certain limitations on the function $h(x)$, $h(0) = 0$ the asymptotic stability of the equilibrium point in the origin for the non-linear system

$$x'(t) = Ax + h(x) \tag{5}$$

$$x(0) = \xi \tag{6}$$

is valid if the matrix $A$ is Hurwitz (has all eigenalues $\lambda$ with $\operatorname{Re}\lambda < 0$).

## 1.6 Inhomogeneous linear systems of ODEs with constant coefficients.

**Corollary. Duhamel formula, autonomous case.** (Corollary 2.17, p. 43)

Consider the inhomogeneous system

$$x'(t) = Ax(t) + g(t)$$

with continuous or piecewise continuous function $g : \mathbb{R} \to \mathbb{R}^N$. Then the unique solution to the I.V.P. with initial data

$$x(0) = \xi$$

is represented by the Duhamel formula:

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))g(\sigma)d\sigma \tag{7}$$

**Proof of the Corollary**: check that the formula gives a solution and show that it is unique.

$$\begin{aligned}
x(t) &= \exp(At)\xi + \int_0^t \exp(A(t-\sigma))g(\sigma)d\sigma \\
&= \exp(At)\xi + \exp(At)\int_0^t \exp(-A\sigma)g(\sigma)d\sigma \\
&= \exp(At)\left[\xi + \int_0^t \exp(-A\sigma)g(\sigma)d\sigma\right] \\
x'(t) &= A\exp(At)\left[\xi + \int_0^t \exp(-A\sigma)g(\sigma)d\sigma\right] + \exp(At)\exp(-At)g(t) \\
&= Ax(t) + g(t)
\end{aligned}$$

for all points $t$ where $g(t)$ is continuous. Difference $z(t) = x(t) - y(t)$ between two solutions $x(t)$ and $y(t)$ satisfies the homogeneous systems $z'(t) = Az(t)$ and zero initial condition $z(0) = 0$ and the integral equation: $z(t) = \int_0^t Az(\sigma)d\sigma$. The same reasoning as before, using the Grönwall inequality, or just a reference to the uniqueness of solutions to homogeneous systems implies that $z \equiv 0$.

## 1.7 Stability of equilibrium points to linear systems perturbed by a small right hand side.

**Theorem** (Theorem 5.27, p. 193, L.R.) **The proof is required at the exam).** Let $G \subset \mathbb{R}^N$ be a non-empty open subset with $0 \in G$. Consider the nonlinear differential equation

$$\begin{aligned}
x'(t) &= Ax(t) + h(x) \qquad (8)\\
x(0) &= \xi \qquad (9)
\end{aligned}$$

where $A \in \mathbb{R}^{N\times N}$ and $h : G \to \mathbb{R}^N$ is a continuous function satisfying

$$\lim_{z\to 0} \frac{h(z)}{\|z\|} = 0. \tag{10}$$

If $A$ is Hurwitz, that is $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$, then 0 is an asymptotically stable equilibrium of 8.

Moreover, there is $\Delta > 0$ and $C > 0$ and $\alpha > 0$ such that for $\|\xi\| < \Delta$ the solution $x(t)$ to the initial value problem with initial data

$$x(0) = \xi$$

exists for all $t \in \mathbb{R}_+$ ad satisfies the estimate

$$\|x(t)\| \leq C \|\xi\| e^{-\alpha t}$$

**Proof. (This proof is required at the exam)**

The main tool in the proof is the following integral form of the I.V.P. based on the Duhamel formula.

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))h(x(\sigma))d\sigma$$

If $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$ then there is $\beta > 0$ such that $\operatorname{Re}\lambda < -\beta$ (strictly smaller!) for all $\lambda \in \sigma(A)$ and

$$\|\exp(At)\| \leq Ce^{-\beta t} \tag{11}$$

for some constant $C > 0$.

We can choose $\varepsilon > 0$ such that $C\varepsilon < \beta$ and using (10) choose $\delta_\varepsilon$ such that for $\|z\| < \delta_\varepsilon$, $z \in G$

$$\frac{\|h(z)\|}{\|z\|} < \varepsilon \tag{12}$$

$$\|h(z)\| < \varepsilon \|z\| \tag{13}$$

It follows from properties of $h$ :$\lim_{z\to 0} \frac{h(z)}{\|z\|} = 0$.

We know from Peano theorem that the solution to the equation (8) exists on some time interval $t \in [0, \delta)$ (another $\delta$!!!)

We apply Duhamel formula (7) for solutions to the equation of interest (8):

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))h(x(\sigma))d\sigma$$

As long as $x(\sigma)$ under the integral, belongs to the ball $\{x : \|x\| < \delta_\varepsilon\} \subset G$, we apply the triangle inequality for integrals and estimates (11) and (12):

$$\begin{aligned}
\|x(t)\| &\leq \|\exp(At)\| \, \|\xi\| + \int_0^t \|\exp(A(t-\sigma))\| \, \|h(x(\sigma))\| \, d\sigma \\
\|x(t)\| &\leq Ce^{-\beta t} \|\xi\| + \int_0^t Ce^{-\beta(t-\sigma)} \varepsilon \, \|x(\sigma)\| \, d\sigma \\
& \quad Ce^{-\beta t} \|\xi\| + e^{-\beta t} \int_0^t C\varepsilon e^{\beta(\sigma)} \, \|x(\sigma)\| \, d\sigma
\end{aligned}$$

Introduce the function $y(t) = \|x(t)\| \, e^{\beta t}$. Then multiplying the last inequality by $e^{\beta t}$

$$\|x(t)\| \, e^{\beta t} \leq C \, \|\xi\| + \int_0^t C\varepsilon \left( \|x(\sigma)\| \, e^{\beta(\sigma)} \right) d\sigma$$

we arrive to

$$y(t) \leq C \, \|\xi\| + \int_0^t (C\varepsilon) \, y(\sigma) d\sigma$$

The Grönwall inequality implies that

$$\|y(t)\| \leq C \, \|\xi\| \, e^{(C\varepsilon)t}$$

and multiplying back with $e^{-\beta t}$

$$\begin{aligned}
\|x(t)\| &\leq C \, \|\xi\| \, e^{-(\beta - C\varepsilon)t} && (14) \\
\|x(t)\| &\leq C \, \|\xi\| \, e^{-\alpha t} && (15)
\end{aligned}$$

It is valid as long as $\|x(t)\| < \delta_\varepsilon$!!!

Now we can choose $\alpha = \beta - C\varepsilon > 0$, by choosing $\varepsilon$ small enough, $\Delta = \min\left\{\frac{1}{2}\delta_\varepsilon/C, \delta_\varepsilon/2\right\}$ and $\|\xi\| < \Delta$. This choice of initial conditions implies that

$$\|x(t)\| \leq \delta_\varepsilon, \tag{16}$$

as long as this solution exists (!!!)

The last estimate (16) implies an important conclusion that the solution can be extended in fact to the whole $\mathbb{R}_+$, because supposing the opposite, namely that there is some maximal existence time $t_{\max} < \infty$, leads to a contradiction.

The fact that solution $x(t)$ can be extended to the whole $\mathbb{R}_+$ and satisfies the estimate

(16) implies that this solution must satisfy the desired estimate

$$\|x(t)\| \leq C \|\xi\| e^{-\alpha t}$$

for all $t \in [0, \infty)$ and implies the asymptotic stability of the equilibrium point in the origin.■

**Black square is for the end of a proof!**

For the extension result above we can refer here to **Lemma 4.9. p. 110.** "On the extension to the boundary point of the open existence time interval for a bounded solution having the closure of the orbit in $G$ " and to the **Corollary 4.10, p. 111.** about "Eternal life" of solutions with orbits enclosed in a compact.

**Lemma 4.9. p. 110.** states that if the positive orbit $O_+$ of a solution $x : (a, b) \longrightarrow R^n$ to the problem:

$$\begin{aligned} x' &= f(t, x) \\ x(\tau) &= \xi \end{aligned}$$

$O_+ = \{x(t) : t \in [\tau, b)\}$ has a compact closure, then this solution can be extended to the closed interval $[\tau, b]$. Applying Peano theorem to the same equation with inital data $(b, x(b))$ - we get an extension to a longer time interval $[\tau, b+\delta)$. Repeating this argument we get an extension to the whole $[\tau, \infty)$.

**Corollary 4.10, p. 111.** considers a maximal solution $x : I_{\max} \rightarrow G$ having it's "future" half - orbit $O_+ = \{x(t) : t \in I_{\max} \cap [\tau, \infty)\}$ contained in a compact subset $C$ : $O_+ \subset C$ of $G$. The conclusion of the Corollary is that such solution must have $I_{\max}$ infinite to the right (in future): $\sup(I_{\max}) = \infty$ for the equation defined on the infinite time interval $[\tau, \infty)$ valid for autonomous ODE here.

We give here a proof of the extension of solution $x(t)$ to $[0, \infty)$, in this particular case, but do not require it at the exam.

**(It is an important theoretical argument. Check similar argument in Lemma 4.9, p. 110 in LR that we formulate in more general situation later )**

The last estimate $\|x(t)\| \leq \delta_\varepsilon$, implies in fact an important conclusion that the solution must exist in fact on the whole $\mathbb{R}_+$, because supposing the opposite, namely that there is some maximal existence time $t_{\max}$, leads to a contradiction.

Let consider this important argument. It consists of two steps.

1) We use the continuity and boundedness of the solution $x(t)$ on $[0, t_{\max})$ together with the integral form of the equation

$$x(t) = \xi + \int_0^t Ax(\sigma)d\sigma + \int_0^t h(x(\sigma))d\sigma$$

The set $\{x(t) : t \in [0, t_{\max})\}$ (that is the orbit of the solution), is bounded according to (16). The closure $\mathfrak{C}$ of this set is therefore compact. The function $h(x)$ is continuous on $G$ and is therefore bounded on the compact set $\mathfrak{C}$.

For any sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to t_{\max}$ the sequence of values $\{x(t_k)\}_{k=1}^{\infty}$is a Cauchy sequence and therefore has a limit

$$\lim_{k\to\infty} x(t_k) = \eta$$

because

$$\begin{aligned} \|x(t_m) - x(t_k)\| &\leq \left\| \int_{t_k}^{t_m} Ax(\sigma)d\sigma + \int_{t_k}^{t_m} h(x(\sigma))d\sigma \right\| \leq \\ \left| \int_{t_k}^{t_m} \|A\| \, \|x(\sigma)\| \, d\sigma \right| + \left| \int_{t_k}^{t_m} \|h(x(\sigma))\| \, d\sigma \right| &\leq C\, |t_m - t_k| \to 0, \quad m, k \to \infty \end{aligned}$$

This limit is unique and independent of the sequence $\{t_k\}_{k=1}^{\infty}$ by a similar estimate. Therefore we can extend $x(t)$ up to the point $t_{\max}$ as

$$x(t_{\max}) \overset{def}{=} \eta = \lim_{t\to t_{\max}} x(t)$$

The extended function $x(t)$ will be continuous on $[0, t_{\max}]$.

2) Now using an existence theorem (Peano or Picard-Lindelöf) for non-linear systems of ODEs, we conclude that there is a solution $y(t)$ to the equation

$$y'(t) = Ay + h(y)$$

on the time interval $[t_{\max}, t_{\max} + \delta)$ with the initial condition $y(t_{\max}) = \eta = x(t_{\max})$ at time $t_{\max}$. This solution is evidently an extension of the original solution $x(t)$ to a larger time interval, that contradicts the our supposition.

Therefore the solution $x(t)$ can be extended to the whole $\mathbb{R}_+$ and satisfies the estimate (16).■

The last theorem implies immediately the following result on the stability of equilibrium points by linearization.

**Theorem.On stability of equilibrium points by linearization.** (Corollary 5.29, p. 195)

Let $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ be a non empty open set with $0 \in G$ , $f$ be continuous and $f(0) = 0$. Let $f$ be differentiable in 0 and $A$ be the Jacoby matrix of $f$ in the point 0: $A = D(f)(0)$:

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(0), \quad i, j = 1, ...N$$

If $A$ is a Hurwitz matrix (all eigenvalues $\lambda \in \sigma(A)$ have $\operatorname{Re}\lambda < 0$), then the equilibrium point of the system

$$x'(t) = f(x(t))$$

in the origin is asymptotically stable.

**Proof.** Consider the function $h(z) = f(z) - Az$. Then by the definition of Jacoby matrix $\|h(z)\| / \|z\| \to 0$ as $z \to 0$. An application of the theorem about stability of a small perturbation of a linear system to the function $f(z) = Az + h(z)$ proves the the claim. ■

The following general theorem by Grobman and Hartman that we formulate without proof is a strong result on connection between solutions to a nonlinear system

$$\begin{aligned} x'(t) &= f(x(t)), & (17) \\ x(0) &= \xi & (18) \end{aligned}$$

with right hand side $f(x)$ close to an equilibrium point $x_*$, $f(x_*) = 0$ and solutions to the linearized system

$$\begin{aligned} y'(t) &= Ay & (19) \\ y(0) &= \xi - x_* & (20) \end{aligned}$$

with constant matrix $A$ that is Jacobi matrix of the right hand side $f$ in the equilibrium point $x_*$, $A = D(f)(x_*)$:

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(x_*), \quad i, j = 1, ...N$$

■

**Definition.** An equilibrium point $x_*$ of the system (17) is called hyperbolic if for all eigenvalues $\lambda \in \sigma(A)$ it is valid that $\operatorname{Re}\lambda \neq 0$.

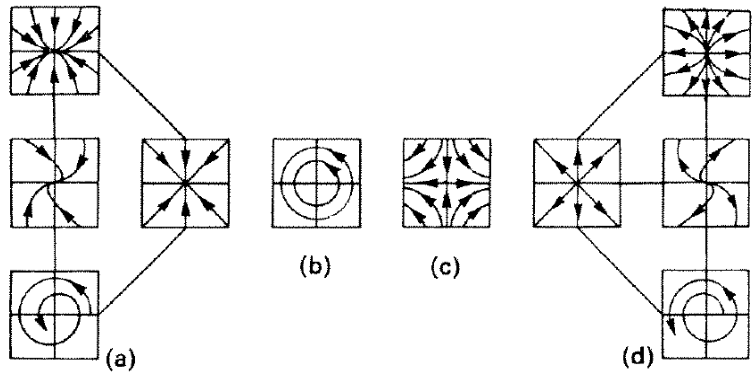
**Theorem. (Grobman-Hartman) A formulation and a (difficult!) proof can be found as Th. 9.9 at the page 266, in the book by Teschl: http://www.mat.univie.ac.a ode/index.html**

Consider an I.V.P. for a autonomous system of differential equations

$$\begin{aligned} x'(t) &= f(x(t)), & (21) \\ x(0) &= \xi & (22) \end{aligned}$$

Let $f \in C^1(B)$, in $B_R(x_*) = \{\xi : \|\xi - x_*\| < R \quad \} \subset G$ and $x_* \in G$ be a hyperbolic equilibrium point of (17): $f(x_*) = 0$.

Then there are neighborhoods $U_1(x_*)$ and $U_2(x_*)$ of $x_*$ and an invertible continuous mapping $R : U_1(x_*) \to U_2(x_*)$ such that $R$ maps shifted solutions $x_* + e^{At}(\zeta - x_*)$ to the




linearized system (19) onto solutions $x(t) = \varphi(t, R(\zeta))$ of the non-linear system (17) with initial data

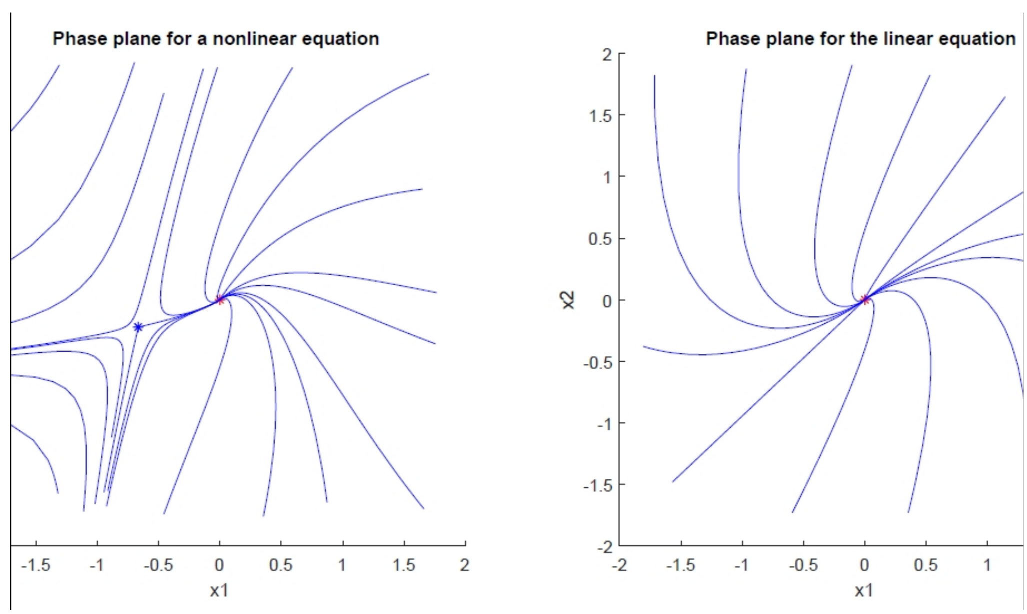
$$\xi = R(\zeta), \zeta = R^{-1}(\xi)$$

$$R\left(x_* + e^{At}(\zeta - x_*)\right) = \varphi(t, R(\zeta))$$

and back

$$R^{-1}(\varphi(t, \xi)) = x_* + e^{At}(R^{-1}(\xi) - x_*)$$

as long as $x_* + e^{At}(R^{-1}(\xi) - x_*) \in U_1(x_*)$.□




Various classes of topologically equivalent equilibrium points in the plane: a) asymptotically stable, b) center, c) saddle point, d) unstable:

In higher dimensions there is a larger variety of topologically different configurations of phase protraits around equilibrium points.

**Example on application of the Grobman - Hartman theorem**

Consider the system

$$x_1' = -\tfrac{1}{2}(x_1 + x_2) - x_1^2$$
$$x_2' = \tfrac{1}{2}(x_1 - 3x_2)$$

It has two equilibrium points: one in the origin $(0,0)$ and the second one is $(-2/3,-2/9)$.We find them by expressing $x_1 = 3x_2$ , from the equation $\frac{1}{2}(x_1 - 3x_2) = 0$, substituting to the equation $-\frac{1}{2}(x_1+x_2) - x_1^2 = 0$, and solving the quadratic equation $-\frac{1}{2}(3x_2+x_2) - 9x_2^2 = 0$ for $x_2$.

$$-\frac{1}{2}(3x_2 + x_2) - 9x_2^2 = -x_2\,(9x_2 + 2) = 0.$$

and its linearization in the origin:

$$x_1' = -\tfrac{1}{2}(x_1 + x_2)$$
$$x_2' = \tfrac{1}{2}(x_1 - 3x_2)$$

The linearized system has matrix $A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$, characteristic polynomial: $\lambda^2 + 2\lambda + 1 = 0$, eigenvalues: $\lambda_{1,2} = -1$. The only eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The origin is a stable for both systems. This equilibrium point is asymptotically stable.

On the other hand we see that another equilibrium $(-2/3,-2/9)$ of the non-linear system seems to be a saddle point.

We check it now. For an arbitrary point we need first to calculate the Jacoby matrix of the right hand side in the system $x' = f(x)$ in an arbitrary point $x \in \mathbb{R}^2$

$$\begin{aligned} [Df]_{ij}(x) &= \frac{\partial f_i}{\partial x_j}(x) \\ [Df](x) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{bmatrix} = \begin{bmatrix} -1/2 - 2x_1 & -1/2 \\ 1/2 & -3/2 \end{bmatrix} \end{aligned}$$

Calculating the Jacoby matrix in the second equilibrium point $(-2/3,-2/9)$ we get the matrix for the linearization of the right hand side in this point:

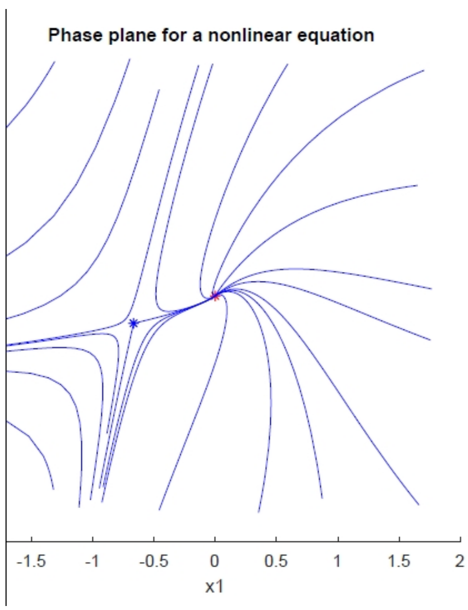
$$A = \begin{bmatrix} -1/2 - 2(-2/3) & -1/2 \\ 1/2 & -3/2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

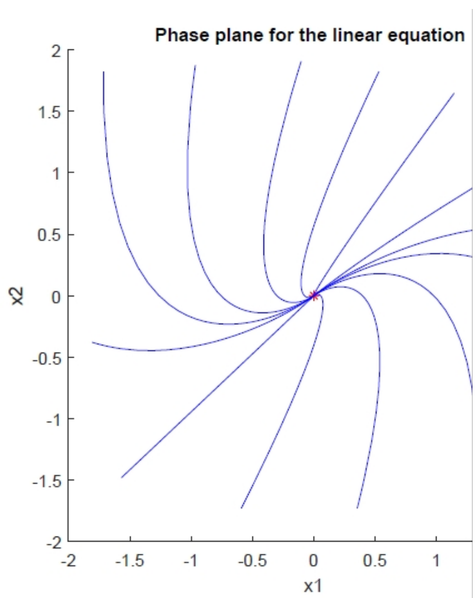
The characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A)$. $\mathrm{tr}(A) = 5/6 - 3/2 = -\frac{2}{3}$. $\det(A) = \frac{5}{6}\left(-\frac{3}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right) = -1$. Therefore $p(\lambda) = \lambda^2 + \frac{2}{3}\lambda - 1$. Eigenvalues are real and have different signs because the determinant $\det(A)$ of $A$ is negative. We do not need to calculate them to make these conclusions.

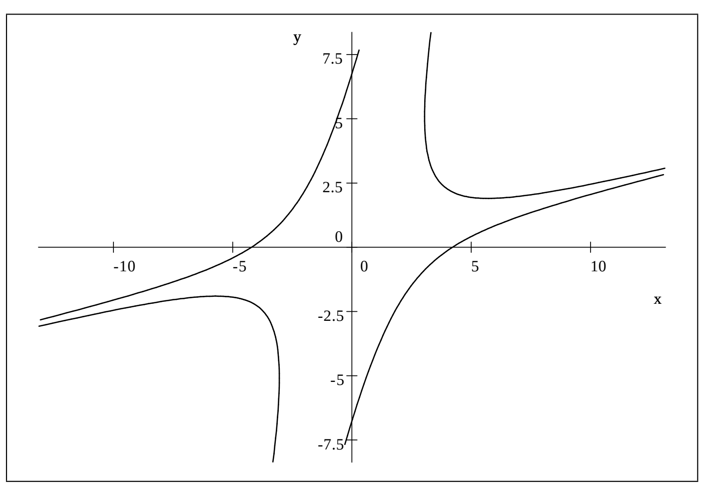
Therefore the linearized system

$$y' = Ay$$

has a saddle point in the origin. The non-linear system also has a saddle point configuration in the phase portrait close to the equilibrium point $(-2/3, -2/9)$ according to the Grobman-Hartman theorem. This equilibrium point is unstable. If we like to sketch a more precise phase portrait for the linearized system we can calculate eigenvalues and eigenvectors. But we can only guess the global phase portrait for the non-linear system (how local phase portraits connect with each other). We give below phase portraits for the non-linear system and for the linearized system around each of equilibrium points.




Phase plane for the linearized system around the equilibrium point $(-2/3, -2/9)$

**Counterexample to the Grobman - Hartman theorem.**

A system such that the linearized system has a center (stable) but the non-linear has an unstable equilibrium point.

Consider the system

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 + (x_1^2 + x_2^2)x_1 \\ \frac{dx_2}{dt} &= -x_1 + (x_1^2 + x_2^2)x_2\end{aligned}$$

The origin $(0,0)$ is an equilibrium point and the linearized system in this point has the form

$$x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

The origin is a center that is a stable equilibrium point.

Consider the equation for $r^2(t) = x_1^2(t) + x_2^2(t)$.We derive it by multiplying the first equation by $x_1$ and the second by $x_2$ and considering the sum of the equations leading to

$$x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} = \frac{1}{2}\frac{d\,(x_1)^2}{dt} + \frac{1}{2}\frac{d\,(x_2)^2}{dt} = \frac{1}{2}\frac{d}{dt}\left(r^2(t)\right) = \left(r^2(t)\right)^2$$

We see that the solution to this equation $z = r^2$

$$\begin{aligned}\frac{1}{2}\frac{dz}{dt} &= z^2 \\ \frac{dz}{z^2} &= 2dt \\ \int \frac{dz}{z^2} &= \int 2dt \\ -\frac{1}{z} &= 2t + C \\ \frac{-1}{z(0)} &= C \\ -\frac{1}{z} &= 2t + \frac{-1}{z(0)} \\ z &= r^2\end{aligned}$$

with separable variables with arbitrary initial data $r(0)$ is

$$r^2(t) = \frac{r^2(0)}{1 - 2r^2(0)t}$$

The solution $r^2(t)$ is increasing with time and tends to infinity with $t$ rising and blows up in finite time.

The equilibrium $(0,0)$ to the nonlinear system is unstable. The phase portraits of the nonlinear system and the linearized system are qualitatively different in this example when eigenvalues to the Jacoby matrix of the right hand side of the nonlinear system in the equilibrium point have real parts equal to zero.

**Example on application of the Grobman - Hartman theorem**

Find for which values of the parameter $a$ the origin is an asymptotically stable equilibrium, stable equilibrium, unstable equilibrium of the following system:

$$\begin{cases} x' = y \\ y' = -ay - x^3 - a^2 x \end{cases} \qquad \textbf{(4p)}$$

**Solution.** Consider the Jacoby matrix of the right hand side in the equatiuon.

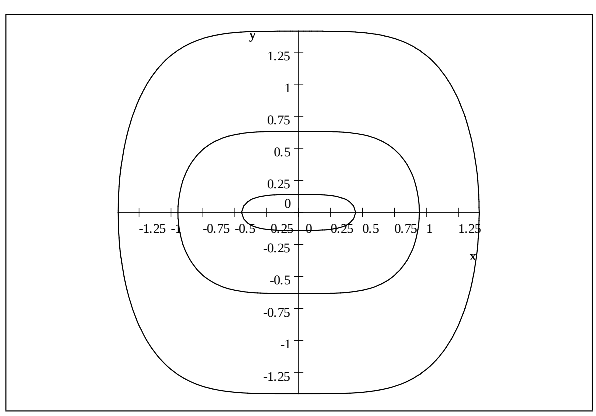
$A(x,y) = \begin{bmatrix} 0 & 1 \\ -a^2 - 3x^2 & -a \end{bmatrix}$. It's value in the origin is $A(0,0) = \begin{bmatrix} 0 & 1 \\ -a^2 & -a \end{bmatrix}$, with characteristic polynomial: $p(\lambda) = \lambda^2 + a\lambda + a^2$.

Eigenvalues are $\lambda_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - a^2} = -\frac{a}{2} \pm i\sqrt{\frac{3a^2}{4}}$ komplex eigenalues with real part $\operatorname{Re}\lambda_{1,2} = -\frac{a}{2}$.

The Grobman - Hartman theorem about stability by linearization imples that the origin is asymptotically stable when $a > 0$ and is unstable when $a < 0$.

For $a = 0$ **linearization does not give any information about stability** because in this case $\operatorname{Re}\lambda_{1,2} = 0$. In this case the system is reduced to $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x^3 \end{cases}$ and we can find an equation for orbits (traces of solutions on the phase plane) of the system from an ODE with separable variables:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{-x^3}{y} \\ ydy &= -x^3 dx \\ \int ydy &= -\int x^3 dx \\ \frac{y^2}{2} &= -\frac{x^4}{4} + C \\ \frac{x^4}{4} + \frac{y^2}{2} &= C \end{aligned}$$




Solutions to the ODE in the case when $a = 0$ will be periodic and go along flattened ellipses in the picture.

**Example. Stability by linearization for the pendulum with friction.**

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{aligned}$$

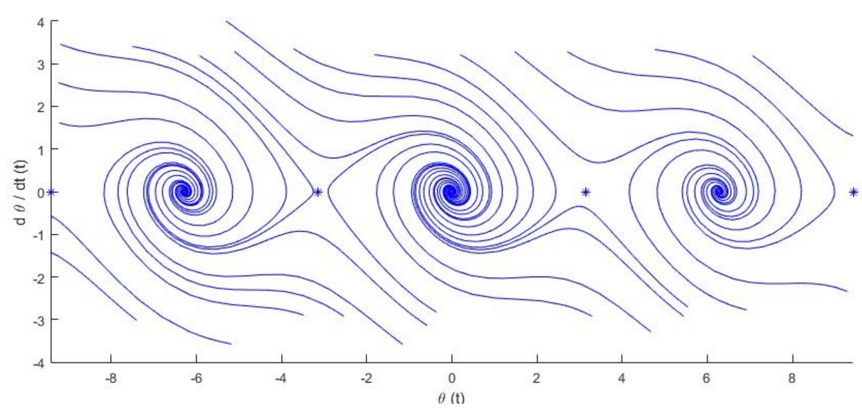
Linearized equation around $(0,0)$ is

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}x_1(t) \end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

$\mathrm{tr}(A) = -\frac{\gamma}{m} < 0$; $\det(A) = \frac{g}{l} > 0$. Therefore the $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(A)$. For small friction coefficient $\gamma$ the equilibrium will be focus, for large friction it will be a stable node. An intermediate case with stable improper node is also possible.




Point out that the case with zero friction: $\gamma = 0$ cannot be treated by linearization, because the linearized system has a center in the origin. The non-linear system has in fact also a center in the origin, but we cannot prove it by means of linearization. We will consider this case later by different means.

**The linearization of the equation around** $(\pi, 0)$.

Linear approximation for sin around $\pi$. Let $(x_1 - \pi) = y_1(t)$.

$$\sin(x_1) = \sin(\pi) + \cos(\pi)(x_1 - \pi) + O(x_1 - \pi)^2 \approx -(x_1 - \pi) = -y_1(t)$$

$$y_1(t) = x_1(t) - \pi; y_1'(y) = x_1'(t)$$

therefore

$$\begin{aligned} x_1(t) &= y_1(t) + \pi;\ x_1'(y) = y_1'(t) \\ x_2(t) &= x_1' = y_1'(t) \end{aligned}$$

Introducing $y_2 = y_1' = x_2;$ we get $x_2 = y_2$

$$\sin(x_1) = \sin(\pi) + \cos(\pi)y_1 + O(\pi - x_1)^2$$

;

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1) \end{aligned}$$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= -\frac{\gamma}{m}y_2(t) - \frac{g}{l}(-y_1) \end{aligned}$$

The linearized equation around $(\pi, 0)$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= \frac{g}{l}y_1(t) - \frac{\gamma}{m}y_2(t) \end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

Characteristic polynomial: $p(\lambda) = \lambda^2 - \left(-\frac{\gamma}{m}\right)\lambda - \frac{g}{l}$.

$\mathrm{tr}(A) = -\frac{\gamma}{m} < 0$; $\det(A) = -\frac{g}{l} < 0$. The equilibrium is always a saddle point (instable).

**Example on application of the Grobman - Hartman theorem**

Find all stationary points of the system of ODE $\begin{cases} x' = e^y - e^x \\ y' = \sqrt{3x + y^2} - 2 \end{cases}$ and investigate their stability by linearization.

1. **Solution.**

   We find stationary points by pointing out that the first equation implies $y = x$ and then $\sqrt{3x + x^2} - 2 = 0$ implies $3x + x^2 - 4 = (x+4)(x-1) = 0$ and therefore two roots $x_1 = 1$ and $x_2 = -4$ follow.

   We have two stationary points: $(1, 1)$ and $(-4, -4)$.

   The Jacobi matrix is $J(x, y) = \begin{bmatrix} -e^x & e^y \\ \frac{3}{2\sqrt{3x+y^2}} & \frac{y}{\sqrt{3x+y^2}} \end{bmatrix}$

   $J(1,1) = \begin{bmatrix} -e & e \\ \frac{3}{2\sqrt{3+1}} & \frac{1}{\sqrt{3+1}} \end{bmatrix} = \begin{bmatrix} -e & e \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$ The trace of $J(1,1)$ is $\operatorname{tr}(J(1,1)) = 1/2 - e < 0$

   $\det(J(1,1)) = e(-1/2 - 3/4) = -\frac{5}{4}e < 0$ it implies that the stationary point $(1,1)$ is has one negative and one postive eigenvalue and therefore is a saddle point and is unstable by the Grobman Hartman theorem.

   The characteristic equation for a 2x2 matrix $A$ is $\lambda^2 - tr(A)\lambda - \det(A) = 0$.

   In this particular situation it is $\lambda^2 + \left(e - \frac{1}{2}\right)\lambda - \frac{5}{4}e = 0$.

   Eigenvalues are: $\lambda_1 = -\frac{1}{2}e + \frac{1}{4} - \frac{1}{4}\sqrt{16e + 4e^2 + 1}$, $\lambda_2 = -\frac{1}{2}e + \frac{1}{4} + \frac{1}{4}\sqrt{16e + 4e^2 + 1}$.

   $J(-4,-4) = \begin{bmatrix} -e^{-4} & e^{-4} \\ \frac{3}{4} & \frac{-4}{2} \end{bmatrix} = \begin{bmatrix} -e^{-4} & e^{-4} \\ \frac{3}{4} & -2 \end{bmatrix}$.

   The trace of $J(-4,-4)$ is $\operatorname{tr}(J(-4,-4)) = -2 - e^{-4} < 0$.

   $\det(J(-4,-4)) = e^{-4}\left(2 - \frac{3}{4}\right) = \frac{5}{4}e^{-4} > 0$. Therefore the the real parts of eigenvalues are negative and the stationary point $(-4,-4)$ is an asymptotically stable equilibrium by the Grobman Hartman theorem.

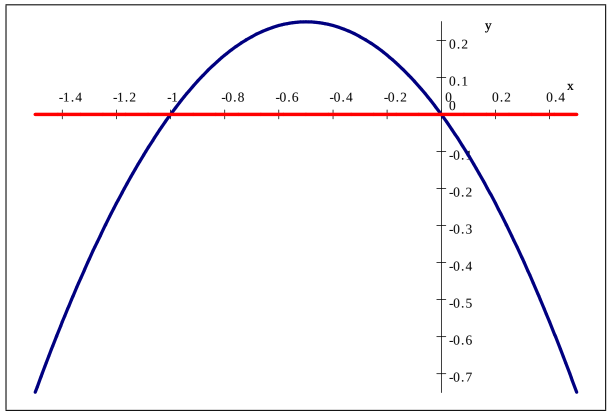
   The characteristic equation is $\lambda^2 + (e^{-4} + 2)\lambda + \frac{5}{4}e^{-4} = 0$.

   Eigenvalues are : $\lambda_1 = -\frac{1}{2}e^{-4} - 1 - \frac{1}{2}\sqrt{\frac{1}{e^8} - \frac{1}{e^4} + 4}$, $\lambda_2 = -\frac{1}{2}e^{-4} - 1 + \frac{1}{2}\sqrt{\frac{1}{e^8} - \frac{1}{e^4} + 4}$

**Example on the application of the Grobman - Hartman theorem**

$$\begin{cases} x' = & y \\ y' = & -y - x - x^2 \end{cases}$$

Nullclines of this system have equations $y = 0$ ( $x$ - nullcline), and $y = -x - x^2$




($y$ - nullcline). Draw directions of velocities on nullclines!

Equilibrium points are intersection popints of different nullclines: $(0,0)$ and $(-1,0)$.

Jacobi matrix of the right hand side in the ODE is $A(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2x & -1 \end{bmatrix}$.

Jacobi matrix in the origin is $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda + 1$, eigenvalues are $-\frac{1}{2}i\sqrt{3} - \frac{1}{2}, \frac{1}{2}i\sqrt{3} - \frac{1}{2}$. Real parts of eigenvalues are negative and therefore the origin is stable focus, asymptotically stable equilibrium.

Jacobi matrix in the point $(-1,0)$ is $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 + \lambda - 1$,eigenvalues are $-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2}$. One is negative, another is positive, the equilibrium point is a saddle point and is unstable. One can also just point out that $\det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = -1 < 0$ that implies the same conclusion using Poincare diagram without calculating eigenvalues.