1 Pendulum without friction. First integral.

$$
\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{g}{l}\sin(x_1(t))
\end{aligned}
$$

2 Stability by linearization for the pendulum with friction.

$$
\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1(t))
\end{aligned}
$$

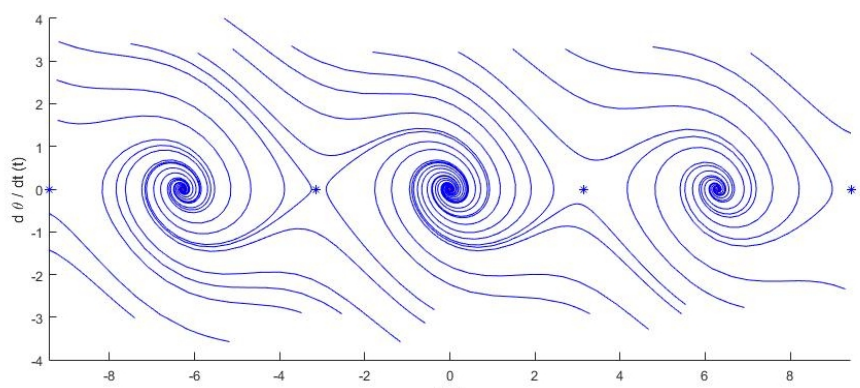
Linearized equation around $(0,0)$ is

$$
\begin{aligned}
x_1'(t) &= x_2(t) \\
x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}x_1(t)
\end{aligned}
$$

The matrix of the system is

$$
A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}
$$

$tr(A) = -\frac{\gamma}{m} < 0$; $\det(A) = \frac{g}{l} > 0$. Therefore the $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$. For small friction coefficient γ the equilibrium will be focus, for large friction it will be a stable node. An intermediate case with stable improper node is also possible.



Point out that the case with zero friction: $\gamma = 0$ cannot be treated by linearization, because the linearized system has a center in the origin. The non-linear system has in fact also a center in the origin, but we cannot prove it by means of linearization. We will consider this case later by different means.

The linearization of the equation around $(\pi, 0)$.

Linear approximation for sin around π. Let $(x_1 - \pi) = y_1(t)$.

$$\sin(x_1) = \sin(\pi) + \cos(\pi)(x_1 - \pi) + O(x_1 - \pi)^2 \approx -(x_1 - \pi) = -y_1(t)$$

$$y_1(t) = x_1(t) - \pi; y_1'(y) = x_1'(t)$$

therefore

$$\begin{aligned} x_1(t) &= y_1(t) + \pi;\ x_1'(y) = y_1'(t) \\ x_2(t) &= x_1' = y_1'(t) \end{aligned}$$

Introducing $y_2 = y_1' = x_2$; we get $x_2 = y_2$

$$\sin(x_1) = \sin(\pi) + \cos(\pi)y_1 + O(\pi - x_1)^2$$

;

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1) \end{aligned}$$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= -\frac{\gamma}{m}y_2(t) - \frac{g}{l}(-y_1) \end{aligned}$$

The linearized equation around $(\pi, 0)$

$$\begin{aligned} y_1'(t) &= y_2(t) \\ y_2'(t) &= -\frac{\gamma}{m}y_2(t) + \frac{g}{l}y_1 \end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

Characteristic polynomial: $p(\lambda) = \lambda^2 - \left(\frac{g}{l}\right)\lambda + \left(\frac{1}{m}\gamma\right)$.

$tr(A) = -\frac{\gamma}{m} < 0$; $\det(A) = -\frac{g}{l} < 0$. The equilibrium is always a saddle point (unstable).

3 Stability for the pendulum with friction by Lyapunov techniques.

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{aligned}$$

Let $k^2 = \frac{g}{l}$

$$\begin{aligned} \theta' &= \psi \\ \psi' &= -\frac{\gamma}{m}\psi - k^2 \sin\theta \end{aligned}$$

The function $V(\theta, \psi)$

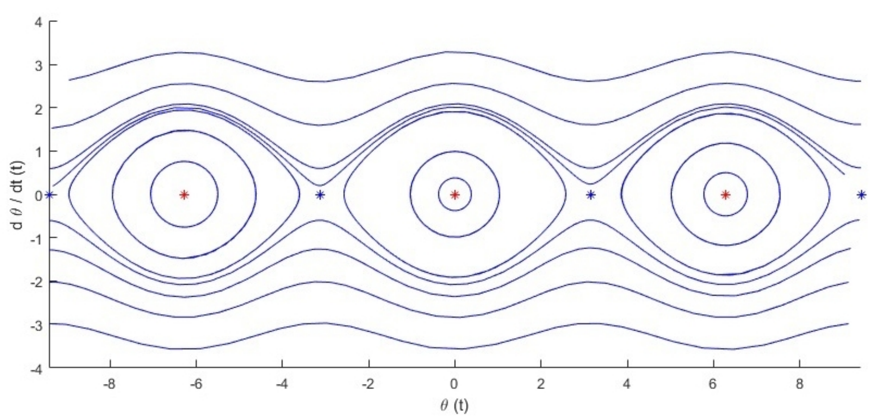
$$V(\theta, \psi) = \frac{\psi^2}{2} + G(\theta)$$

with $G(\theta) = k^2(1 - \cos\theta)$ is the first integral of the system describing the pendulum without friction.

$$\begin{aligned} \nabla V \cdot f &= V_f = \begin{bmatrix} k^2 \sin\theta \\ \psi \end{bmatrix} \cdot \begin{bmatrix} \psi \\ -\left(\frac{\gamma}{m}\psi + k^2 \sin\theta\right) \end{bmatrix} = \\ &= \psi k^2 \sin\theta - \psi k^2 \sin\theta - \left(\frac{\gamma}{m}\right)\psi^2 = -\left(\frac{\gamma}{m}\right)\psi^2 \leq 0 \end{aligned}$$

Level sets of the function $V(\theta, \psi) = h$ consist of the orbits of the system without friction $\gamma = 0$.

$$\begin{aligned} \frac{\psi^2}{2} + G(\theta) &= h \\ \psi &= \pm\sqrt{2\,(h - G(\theta))} = \pm\sqrt{2\,(h - k^2(1 - \cos\theta))} \end{aligned}$$



There are level sets corresponding to $h = 2k^2$ consisting with of upper unstable equilibrium points where $\cos\theta = -1$
and orbits connecting them and corresponding to trajectories that tend to the upper unstable equilibriums and not rotating further. Level sets with $h > 2k^2$ correspond to unbounded trajectories and the rotation of the pendulum around the pivot. Level sets corresponding to $h < 2k^2$ correspond to periodic solutions and surround just one equilibrium point.