First integrals

One dimensional Newton's equation. First integrals

Consider a similar example

$$mx'' + g(x) = 0,$$

$xg(x) > 0$, $x \neq 0$, $g(0) = 0$. Suppose that $\int_0^x g(s)ds \to \infty$ as $x \to \infty$.

It describes a spring with non-linear force $-g(x)$. It can be rewritten as a system of ODE's of the first order.

$$\begin{aligned} x_1 &= x \\ x' &= x_1' = x_2, \\ mx_2' &= -g(x_1) \end{aligned}$$

Consider the test function $V(x_1, x_2)$ in the following form:

$$V(x_1, x_2) = \frac{m}{2}(x_2)^2 + \int_0^{x_1} g(s)ds$$

representing the energy of the system, consisting of two terms: the kinetic energy $\frac{m}{2}(x')^2$ and the potential energy $G(x) = \int_0^x g(s)ds$.Derivatie of the potential energy is equal to the force.

Point out that V is positive definite because of the limitation $xg(x) > 0$, $x \neq 0$.

Consider the derivative V_f of V along trajectories

$$\begin{aligned} (\nabla V \cdot f)(x_1, x_2) &= \left(\frac{\partial}{\partial x_1} V\right) f_1 + \left(\frac{\partial}{\partial x_2} V\right) f_2 \\ &= g(x_1)x_2 + mx_2\left(-\left(\frac{1}{m}\right) g(x_1)\right) = 0 \quad !!!! \end{aligned}$$

We point out also that $(\nabla V \cdot f)(x_1, x_2) = 0$ is zero everywhere and therefore $V(x)$ is constant along trajectories of the system. Such function is called **the first integral** of the system.

Definition

Functions that satisfy the relation $(\nabla V \cdot f)(x_1, x_2) = 0$ and are therefore constant on trajectories of the system $x' = f(x)$ are called *first integrals* of the system.

Property of level sets of first integrals.

Level sets of a first integral V have the property that velocities $f(x)$ of the system are tangent or zero on all level sets of V. It implies that these level sets are unions of orbits of the system.

It lets draw a phase portrait of such system as a number of level sets of the energy V.

We can express level sets

$$V(x_1, x_2) = h$$

of the first integral V in the example above as

$$x_2 = \pm\sqrt{\frac{2}{m}\left(h - G(x_1)\right)}$$

that is valid in points where the expression under the root is non-negative.

Proposition. 4.54, p. 161

If the first integral V has level sets that are closed curves that do not contain equilibrium points, these curves are orbits of periodic solutions.

This idea is almost the only constructive method to calculate periodic orbits for non-linear systems in plane.

Pointing out that $G(x_1) = \int_0^{x_1} g(s)ds$ in the example above is monotone with respect to $|x_1|$, we conclude that those level sets of $V(x_1, x_2)$ that are closed curves and contain no equilibrium points must be orbits of periodic solutions, according to Poincare-Bendixson theorem. It implies in particular that the origin is not asymptotically stable equilibrium point in this example.

1 Stability by linearization for the pendulum with friction.

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{aligned}$$

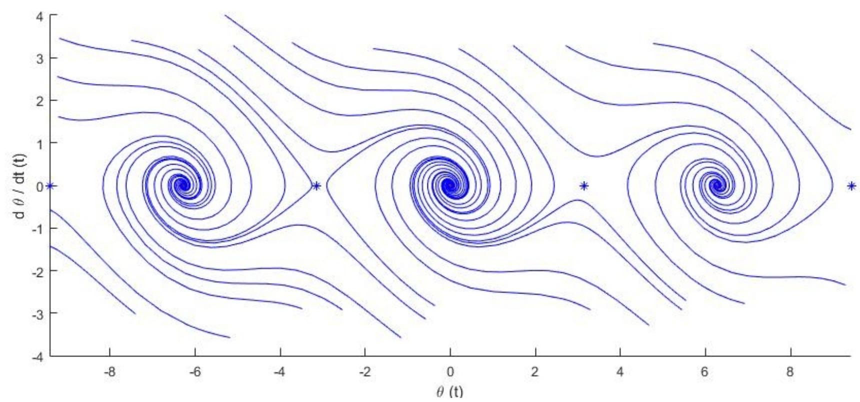
Linearized equation around $(0,0)$ is

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m}x_2(t) - \frac{g}{l}x_1(t) \end{aligned}$$

The matrix of the system is

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{bmatrix}$$

$tr(A) = -\frac{\gamma}{m} < 0$; $\det(A) = \frac{g}{l} > 0$. Therefore the $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$. For small friction coefficient γ the equilibrium will be focus, for large friction it will be a stable node. An intermediate case with stable improper node is also possible.



Point out that the case with zero friction: $\gamma = 0$ cannot be treated by linearization, because the linearized system has a center in the origin. The non-linear system has in fact also a center in the origin, but we cannot prove it by means of linearization. We will consider this case later by different means.

The linearization of the equation around $(\pi, 0)$.

Linear approximation for sin around π. Let $(x_1 - \pi) = y_1(t)$.

$$\sin(x_1) = \sin(\pi) + \cos(\pi)(x_1 - \pi) + O(x_1 - \pi)^2 \approx -(x_1 - \pi) = -y_1(t)$$

$$y_1(t) = x_1(t) - \pi; y_1'(y) = x_1'(t)$$

therefore

$$\begin{aligned} x_1(t) &= y_1(t)+\pi;\ x_1'(y)=y_1'(t)\\ x_2(t) &= x_1'=y_1'(t)\end{aligned}$$

Introducing $y_2 = y_1' = x_2$; we get $x_2 = y_2$

$$\sin(x_1)=\sin(\pi)+\cos(\pi)y_1+O(\pi-x_1)^2$$

;

$$\begin{aligned} x_1'(t) &= x_2(t)\\ x_2'(t) &= -\frac{\gamma}{m}x_2(t)-\frac{g}{l}\sin(x_1)\end{aligned}$$

$$\begin{aligned} y_1'(t) &= y_2(t)\\ y_2'(t) &= -\frac{\gamma}{m}y_2(t)-\frac{g}{l}(-y_1)\end{aligned}$$

The linearized equation around $(\pi,0)$

$$\begin{aligned} y_1'(t) &= y_2(t)\\ y_2'(t) &= -\frac{\gamma}{m}y_2(t)+\frac{g}{l}y_1\end{aligned}$$

The matrix of the system is

$$A=\begin{bmatrix} 0 & 1\\ \frac{g}{l} & -\frac{\gamma}{m}\end{bmatrix}$$

Characteristic polynomial: $p(\lambda)=\lambda^2-\left(\frac{g}{l}\right)\lambda+\left(\frac{1}{m}\gamma\right)$.

$tr(A)=-\frac{\gamma}{m}<0$; $\det(A)=-\frac{g}{l}<0$. The equilibrium is always a saddle point (unstable).

2 Pendulum with and without friction and the first integral.

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{\gamma}{m} x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{aligned}$$

Let $k^2 = \frac{g}{l}$

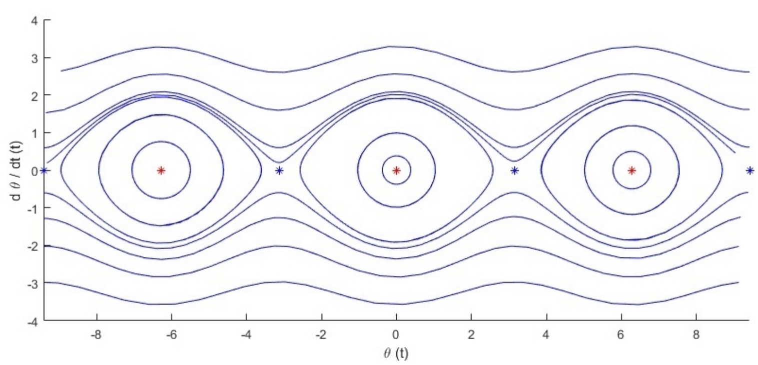
$$\begin{aligned} \theta' &= \psi \\ \psi' &= -\frac{\gamma}{m}\psi - k^2 \sin\theta \end{aligned}$$

The function $V(\theta,\psi) = \frac{\psi^2}{2} + G(\theta)$ with $G(\theta) = k^2(1-\cos\theta)$ is the first integral of the system describing the pendulum without friction.

$$\begin{aligned} \frac{dV}{dt} &= \nabla V \cdot f = \begin{bmatrix} k^2 \sin\theta \\ \psi \end{bmatrix} \cdot \begin{bmatrix} \psi \\ -\left(\frac{\gamma}{m}\psi + k^2 \sin\theta\right) \end{bmatrix} = \\ &= \psi k^2 \sin\theta - \psi k^2 \sin\theta - \left(\frac{\gamma}{m}\right)\psi^2 = -\left(\frac{\gamma}{m}\right)\psi^2 \le 0 \end{aligned}$$

Level sets of the function $V(\theta,\psi) = h$ consist of the orbits of the system without friction $\gamma = 0$.

$$\begin{aligned} \frac{\psi^2}{2} + G(\theta) &= h \\ \psi &= \pm\sqrt{2\,(h - G(\theta))} = \pm\sqrt{2\,(h - k^2(1-\cos\theta))} \end{aligned}$$



Level sets corresponding to $h = 2k^2$ consisting with of upper unstable equilibrium points where $\cos\theta = -1$ and orbits connecting them and corresponding to trajectories that tend to the upper unstable equilibriums and not rotating further.

Level sets with $h > 2k^2$ correspond to unbounded trajectories and the rotation of the pendulum around the pivot.

Level sets corresponding to $h < 2k^2$ correspond to periodic solutions and surround just one equilibrium point.

First integrals for Lotka - Volterra predator prey model

Consider system

$$\begin{aligned} x' &= x - xy \\ y' &= -y + xy \end{aligned}$$

and find its orbits as level sets of a first integral

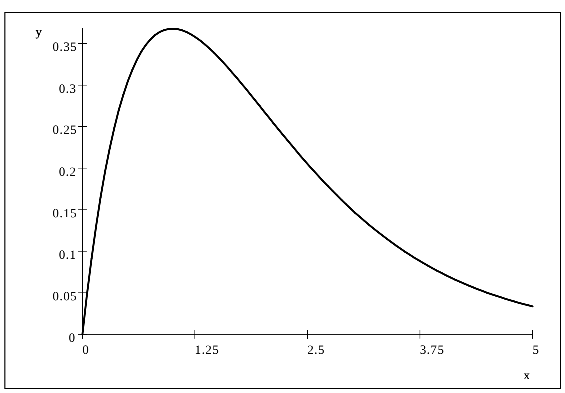
$$\begin{aligned} \frac{dy}{dx} &= \frac{-y + x_1 y}{x - xy} = \frac{y\,(x-1)}{x\,(1-y)} \\ \frac{(1-y)}{y}dy &= \frac{(x-1)}{x}dx \end{aligned}$$

$\int \frac{(x-1)}{x}dx = \int 1 - \frac{1}{x}dx = x - \ln x = -\phi(x)$

$$-\phi(x) = \phi(y) + K$$

$$\begin{aligned} -\phi(x) - \phi(y) &= K \\ -x + \ln x - x + \ln x &= K \\ \exp(\phi(x) + \phi(y)) &= xye^{-x}e^{-y} = C \\ V(x,y) &= g(x)g(y) = C \\ g(x) &= xe^{-x} \end{aligned}$$

$g(x)$ has the only maximum at $x = 1$ and the first integral $V(x,y)$ has a maximum at $(1,1)$.



Therefore there is a neighbourhood of $(1,1)$ where level curves of $V(x,y)$ are closed and therefore the equilibrium $(1,1)$ is a centerand is stable, but not asymptotically stable.

Lyapunov stability theory

The pioneering work by Lyapunov on stability theoy, where both the idea of linearization and the idea of test functions were introduced and developed, was his Ph.D thesis published in 1892 and translated to French in 1907.

(§5.1 in L.R.)

Consider an autonomous system $x' = f(x)$ with $f : G \to \mathbb{R}^N$, $G \subset \mathbb{R}^N$ open. We suppose that f is a locally Lipschitz continuous function, so the existence and uniqueness of maximal solutions to I.V.P. are valid.

We repeat for convenience definitions of stable and unstable equilibrium points

(Equilibrium points are considered here at the origin to make it simpler to apply the construction with Lyapunov functions)

Comment. In fact $\mathbb{R}^+ \subset I_\xi$ in this case.

Definition

An equilibrium point $0 \in G$ of the system $x' = f(x)$ is said to be stable if for each $\varepsilon > 0$, there is $\delta > 0$ such that for any ξ taken in the ball $B(\delta, 0)$=$\{\xi \in \mathbb{R}^N, \quad |\xi| < \delta\}$ the maximal solution $x(t) = \varphi(t, \xi)$: $I_\xi \to G$ on the maximal interval I_ξ with initial data $x(0) = \xi$ and $0 \in I_\xi$ will stay in the ball $B(\varepsilon, 0)$: $\|\varphi(t, \xi)\| < \varepsilon$ for all $t \in I_\xi \cap \mathbb{R}^+$.

Definition

The function $V : U \to \mathbb{R}$, U - open, containing the origin $0 \in U$, is said to be positive definite in U, if $V(0) = 0$ and $V(z) > 0$ for $\forall z \in U$, $z \neq 0$.

Lyapunov's theorem on stability

Theorem. Th.5.2, p.170

Let 0 be an equilibrium point for the system above and there is a **positive definite** continuously differentiable, $C^1(U)$ function $V : U \to \mathbb{R}$, such that $U \subset G$, $0 \in U$ and $V_f(z) = \nabla V \cdot f(z) \leq 0 \; \forall z \in U$.

Then 0 is a stable equilibrium point.

Remark.

A function V with these properties is usually called the **Lyapunov function of the system.**

Proof.

Take an arbitrary $\varepsilon > 0$ such that $B(\varepsilon, 0) \subset U$ and $\partial B(\varepsilon, 0) \subset U$ for $\partial B(\varepsilon, 0) = S(\varepsilon, 0) = \{z : \|z\| = \varepsilon\}$.

Let

$$\alpha = \min_{z \in S(\varepsilon,0)} V(z)$$

be a minimum of the continuous function V on the boundary of $B(\varepsilon, 0)$, that is the sphere $S(\varepsilon, 0) = \{z : |z| = \varepsilon\}$ and is a compact set (closed and bounded).

Then $\alpha > 0$ because $V(z) > 0$ outside the equilibrium point 0.

By continuity of the function V and the fact that $V(0) = 0$ one can find a δ, $0 < \delta < \varepsilon$ such that $\forall z \in B(\delta, 0)$ we have $V(z) < \alpha/2$.

On the other hand for any part of the trajectory $x(t) = \varphi(t, \xi)$, inside U the function $V(\varphi(t, \xi))$ is *non-increasing* because $\frac{d}{dt}V(\varphi(t, \xi)) = (\nabla V \cdot f)(x(t)) \leq 0$.

It implies all trajectories $\varphi(t, \xi)$ with initial points $\xi \in B(\delta, 0)$ satisfy $V(\xi) < \alpha/2$. Therefore $V(\varphi(t, \xi)) < \alpha/2$ and $\varphi(t, \xi)$ cannot reach the sphere $S(\varepsilon, 0)$ where $V(z) \geq \alpha = \min_{z \in S(\varepsilon,0)} V(z)$.

Therefore any such trajectory stays within the ball $B(\varepsilon, 0)$ and by the definition, the equilibrium point in the origin 0 is stable.

It implies also that $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ, because the trajectory stays inside a compact set. ■

Remark. The definition of stability and proofs of the theorems are exactly the same if we take an arbitrary equilibrium point x_0 instead of the origin and use balls $B(\varepsilon, x_0)$ around x_0.

Example.

Investigate stability of the equilibrium point in the origin for the following system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

that follows from the second order equation $x'' + (x')^3 + x = 0$.

Try the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$. It is positive definite.

We check the sign of the derivative of V along trajectories of solutions: $V_f(x_1, x_2) = (\nabla V \cdot f)(x_1, x_2) = 2x_1x_2 + 2x_2(-x_1 - x_2^3) = -4x_2^4 \leq 0$.

Point out that $V_f(x_1, x_2) = 0$ along the x_1 axis where $x_2 = 0$, not only in the origin!!!

Theorem. Asymptotic stability by Lyapunov's functions. Cor. 5.17, p.185,

In the course book this theorem is considered as a corollary to a more general and difficult LaSalle's invariance principle. We give here an independent elementary proof to asymptotic stability.

Let 0 be an equilibrium point for the system above and let V be a positive definite, continuously differentiable function $V : U \to \mathbb{R}$, such that $U \subset G$, U - open, $0 \in U$, and $V_f(z) = \nabla V \cdot f(z) < 0$ (strict inequality outside the equilibrium point!) $\forall z \in U$, $z \neq 0$,

Then 0 is an asymptotically stable equilibrium point.

Definition. Lyapunov functions satisfying conditions in this theorem are often called *strong Lyapunov functions.*

Proof.

By the Lyapunov's stability theorem the origin is a stable equilibrium and therefore for any ball $B(R, 0)$ there is a ball $B(r, 0) \subset U$ such that for any $\xi \in B(r, 0)$, $\varphi(t, \xi) \in B(R, 0)$ for any time $t \in I_\xi$, and $\mathbb{R}^+ \subset I_\xi$, where I_ξ is the maximal interval for initial point ξ.

Therefore we need only to show that the origin is an attractor. Namely we need to show that there is a ball $B(r, 0) \subset U$, such that for any $\xi \in B(r, 0)$ it follows that $\varphi(t, \xi) \to 0$ as $t \to \infty$.

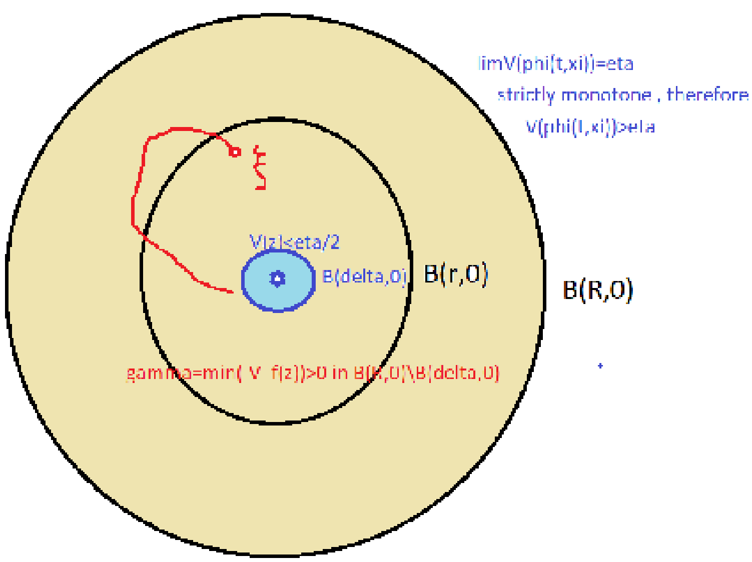
1) It suffices to show that $\lim_{t\to\infty} V(\varphi(t, \xi)) = 0$ because V is continuous and is positive outside the origin, where $V(0) = 0$. It will imply that $\varphi(t, \xi) \to 0$ as $t \to \infty$.

This implication is easy to proof by the following contradiction argument.

Suppose that $\varphi(t, \xi)$ does not tend to the origin for some $\xi \in B(r, 0)$.

Then there is a constant $\varepsilon > 0$ and a sequence of times $t_k \to \infty$ as $k \to \infty$ such that $\|\varphi(t_k, \xi)\| > \varepsilon > 0$.

It implies that $V(\varphi(t_k, \xi)) > q > 0$ for some positive constant q. But it is not compatible with supposition that $\lim_{t\to\infty} V(\varphi(t, \xi) = 0.\square$



2) Now we continue proving that $\lim_{t\to\infty} V(\varphi(t, \xi) = 0$. By conditions of the theorem $\frac{d}{dt} V(\varphi(t, \xi) < 0$, therefore $V(\varphi(t, \xi)$ is a monotone strictly decreasing function of t bounded from below because $V(x) \geq 0$ and must have a limit

$$\lim_{t\to\infty} V(\varphi(t, \xi) = \eta, \quad t \to \infty.$$

We again apply the argument with contradiction.

Suppose that this limit η is not zero: $\eta > 0$. Then $V(\varphi(t,\xi) > \eta > 0$ for all $t \geq 0$ because $V(\varphi(t,\xi)$ is stricly monotone decreasing.

Now we like to find a ball $B(\delta, 0)$, $\delta < r$ around the origin so small that the trajectory $\varphi(t,\xi)$ cannot reach it. The idea is that outside this ball (where our trajectory $\varphi(t,\xi)$ is situated) the decreasing rate for $V(\varphi(t,\xi)$ along the trajectory is never close to zero. This fact would lead us to a contradiction with out supposition.

Continuity of V and the fact that $V(0) = 0$ imply that there is a ball $B(\delta, 0)$, $\delta < r$ such that $0 \leq V(z) < \eta/2$ for all $z \in B(\delta, 0)$.

Hence $\varphi(t,\xi)$ cannot reach it: $\|\varphi(t,\xi)\| \geq \delta$ for all $t \geq 0$, because $V(\varphi(t,\xi) > \eta > 0$ for all $t \geq 0$ by our supposition that $V(\varphi(t,\xi)) \searrow \eta$ as $t \to \infty$.

Now we will estimate the smalles rate of decrease for $V(\varphi(t,\xi)$ that follows from our conclusions. Consider the closed spherical slice $S = \{z : \delta \leq \|z\| \leq R\}$ where the trajectory $\varphi(t,\xi)$ is situated, and point out that $\gamma = \min_{z \in S}(-V_f(z)) > 0$ exists because S is compact and V_f is continuous.

$\gamma > 0$ by the condition of the theorem that $V_f < 0$ outside the origin. Therefore

$$-\frac{d}{dt}V(\varphi(t,\xi)) = -V_f(\varphi(t,\xi)) \geq \gamma = \min_{z \in S}(-V_f(z))$$

and

$$\frac{d}{dt}V(\varphi(t,\xi)) \leq -\gamma$$

By integration from 0 to t we arrive to

$$V(\varphi(t,\xi)) - V(\xi) \leq -\gamma t \to -\infty$$

as $t \to \infty$ that contradicts to the supposition that $V(z) \geq 0$.

It implies that our supposition that $\lim_{t\to\infty} V(\varphi(t,\xi) = \eta > 0$ was wrong and that

$$\lim_{t\to\infty} V(\varphi(t,\xi) = 0$$

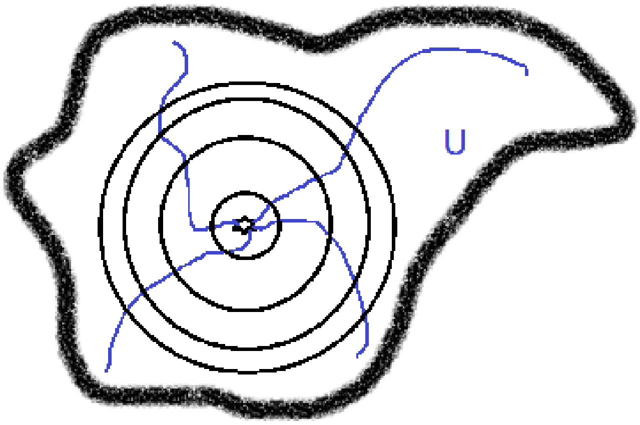
As we pointed out at the beginning of the proof, the last fact implies, that $\lim_{t\to\infty} \varphi(t,\xi) = 0$ and therefore the origin is an attractor and is an asymptotically stable equilibrium point.■

Remark.

This theorem on asymptotic stability has a (very difficult!) inversion (proven in 1949,1956) by José Luis Massera, Uruguay, stating that for any system with an asymptotically stable equilibrium point, there is a "strong" Lyapunov function V such that $V_f(z) < 0$ in a neighborhood of this equilibrium point (outside the equilibrium point $z = 0$ itself).

Definition. Region of attraction for an asymptotically stable equilibrium point.

A domain $U \subset G$ is called the region of attraction for an asymptotically stable equilibrium point $x_* \in U$ if for any $\xi \in U$, the maximal existence interval I_ξ of the the solution $x(t) = \varphi(t,\xi)$ contains $\mathbb{R}^+ \subset I_\xi$ and $\varphi(t,\xi) \to x_*$ as $t \to \infty$.



Example. Consider the system of equations

$$\begin{aligned} x' &= -x + 2xy^2 \\ y' &= -(1-x^2)y^3 \end{aligned}$$

Investigate stability of the equilibrium point in the origin and find possible region of attraction.

Point out that for the right hand side in the equation the Jacoby matrix J in the origin is degenerate $J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ and the linearization of the system does not give any information about stability of the equilibrium in the origin.

Consider the simplest test function $V(x,y) = x^2 + y^2$.

$$\begin{aligned} V_f(x,y) &= (\nabla V \cdot f)(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -x + 2xy^2 \\ -(1-x^2)y^3 \end{bmatrix} \\ &= 2x\left(-x + 2xy^2\right) + 2y\left(-(1-x^2)y^3\right) = 4x^2y^2 - 2y^4 - 2x^2 + 2x^2y^4 \\ &= -2x^2(1-2y^2) - 2y^4(1-x^2) \end{aligned}$$

$V_f(x,y) < 0$ in the rectangle $Q = (-1,1) \times \left(-1/\sqrt{2}, 1/\sqrt{2}\right)$, $(x,y) \neq 0$. Therefore the function V is a strong Lyapunov function and the origin is an asymptotically stable equilibrium.

The region of attraction is bounded by the largest level set of V (circle around the origin) that fits into the rectangle Q: for example $x^2 + y^2 < 1/2$.

This region of attraction reported here is just one we could find using this particular Lyapunov function. It can exist a larger region of attraction that we could not identify by this simple choice of Lyapunov function.

Example. Consider the system of equations

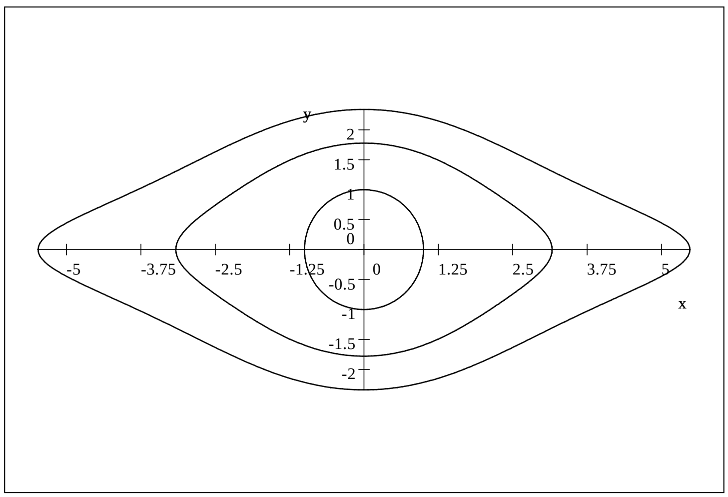
$$\begin{aligned} x' &= -x^3 - 2xy^2 \\ y' &= x^2 y - y^3 \end{aligned}$$

Investigate stability of the equilibrium point in the origin finding a suitable Lyapunov function. It is not trivial task and needs some creative thinkling. Consider the following test function:

$$V(x, y_2) = x^2 + x^2 y^2 + y^4$$

The test function V is positive definite: $V(0,0) = 0$, $V(x,y) > 0$ for $(x,y) \neq (0,0)$.

We draw several level sets for $V(x,y) = x^2 + x^2y^2 + y^4 = h$, for $h = 1, 20, 30$.



We choose the form of the test function in such a way that on the level set curves of this function velocities $f(x,y)$ point strictly inward: $V_f(x,y) = (\nabla V \cdot f)(x,y) < 0$. We have chosen the term y^4 having $\frac{\partial}{\partial y_2}(y^4) = 4y^3$ that after multiplication with the term $-y^3$ from f_2 gives a "good" negative term $-4y^6$. Similarly $\frac{\partial}{\partial y}(y^2) = 2y$ multyplied by the term $-y^3$ from f_1 gives a good negative term $-2y^4$.

The tricky step is to play with "bad" indefinite mixed products in such a way that they (in the best case!) would give no terms in $(\nabla V \cdot f)(x,y)$ with indefinite sign or at least pop up with small coefficients

.

$$
\begin{aligned}
V_f(x,y) &= (\nabla V \cdot f)(x,y) = \\
&\left(2x + 2xy^2\right)\left(-x^3 - 2xy^2\right) + \\
&+ \left(2x^2y + 4y^3\right)\left(x^2y - y^3\right) \\
&= -2x^4 - 4x^2y^2 - \widehat{2x^4y^2} - \underbrace{4x^2y^4} \\
&\quad + \widehat{2x^4y^2} - 2x^2y^4 + \underbrace{4x^2y^4} - 4y^6 \\
&= -2x^4 - 4y^6 - 4x^2y^2 - 2x^2y^4 \\
&= \left(-x^4 - 2y^6 - 2x^2y^2 - x^2y^4\right)2 \\
&\leq \left(-x^4 - 2y^6\right)2 < 0, \quad (x,y) \neq (0,0)
\end{aligned}
$$

Therefore according to the last theorem, the origin $(0,0)$ is an asymptotically stable equilibrium point. The test function tends to inifinity with $\|(x,y)\| \to \infty$. It implies that the equilibriuh has the whole plane $\mathbb{R}^2$ as the region of attraction. All trajectories $\varphi(t,\xi)$ tend to the origin with $t \to \infty$: $\varphi(t,\xi) \underset{t\to\infty}{\longrightarrow} (0,0)$.

Remark

One can arrive to indefinite terms after calculation of $V_{f.}$ .It is still not the end of the hope. One can check that these indefinite terms are not large comparing with negative definite ones and might be compensated by negative definite terms in the expression for $V_{f.}$. For example the expression $-x^2 - y^2 + xy < 0$ for $(x, y) \neq (0, 0)$ because $2|xy| \leq x^2 + y^2$. One can also use known criteria for positive and negative definite quadratic forms in such problems.

The following two inequalities are useful tools for many problems in analysis.

Cauchy inequality:

$$|x|\,|y| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

Young inequality. If $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for every pair of numbers $p, q \in (1, \infty)$ satisfying the conjugacy relation.

$$\frac{1}{p} + \frac{1}{q} = 1$$

The Cauchy inequality is the simplest example of the Young inequality for $p = q = 2$.

Example 8. This example demonstrates how to use Cauchy and Young inequalities for estimating $V(x, y)$ and $V_f(x, y)$.

Two following inequalities are convenient to estimate absolute value of "bad" indefinite terms like xy, x^3y^2, xy^2, et.c. giving possibility to observe that "bad terms" are dominated by positive definite terms.

Example.

Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for it.

Hint: applying Lyapunovs theorem with a simple Lyapunov function, use the Cauchy inequality $2xy \leq (x^2 + y^2)$ to estimate indefinite terms with xy.

Solution. Choose a test function $V(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$

$$\begin{aligned} V_f &= x(-x - 2y + xy^2) + y\left(3x - 3y + y^3\right) = xy - x^2 - 3y^2 + y^4 + x^2y^2 = \\ &= -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + xy \leq 0 \quad ????? \end{aligned}$$

We apply the Cauchy inequality $|x|\,|y| \leq \frac{1}{2}\left(x^2 + y^2\right)$ to the last indefinite term xy and collecting terms with x^2 and y^2 and arrive to the estimate

$$V_f \leq -x^2\left(0.5 - y^2\right) - y^2\left(2.5 - y^2\right)$$

It implies that $V_f(x,y) < 0$ for $(x,y) \neq (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the Lyapunov function is strong and the origin is asymptotically stable.

The attracting region is bounded by the largest level set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely

$(x^2 + y^2) < 1/2$.

Another more clever choice of a test function is $V(x,y) = 3x^2 + 2y^2$.

$$\begin{aligned} V_f &= 6x(-x - 2y + xy^2) + 4y(3x - 3y + y^3) = 4y^4 - 12y^2 - 6x^2 + 6x^2y^2 = \\ &= -4y^2\left(3 - y^2\right) - 6x^2\left(1 - y^2\right) < 0 \end{aligned}$$

for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ is a region of attraction for the asymptotically stable equilibrium in the origin.

Lyapunov's theorem on instability

We give here a slightly weaker variant of the instability theorem comparing with one in the book. An advantage of the variant here is that it suggests a more constructive proof.

Students are free to choose any of these two variants at the examination.

Definition.

An equilibrium point $0 \in G$ of the system is said to be unstable if it is not stable.

Explicit version of the same definition.

There is a ball $B(R,0) \subset G$ such that for any $\delta > 0$ there is a point $\xi \in B(\delta, 0)$ such that for the trajectory $\varphi(t,\xi)$ starting in ξ there is time $t_* \in I_\xi$ such that $\varphi(t,\xi) \notin B(R,0)$.

Another reformulation of this definition is possible.

Another explicit version of the same definition

There is a ball $B(R,0) \subset G$ and a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = 0$ such that for each maximal solution $\varphi(t,\xi)$ with initial data $\xi = x_n$ there is time $t_* \in I_\xi$ such that $\varphi(t,\xi) \notin B(R,0)$.

Theorem. On a criterium of instability of an equilibrium using test functions.

Let the origin 0 - be the equilibrium point of the system $x' = f(x)$. Assume that there is a neighbourhood $U \subset G$, $0 \in U$ and a continuously differentible $C^1(U)$ function $V : U \to \mathbb{R}$ satisfying the following hypotheses.

1) $V_f(z) = \nabla V \cdot f(z) > 0$ for every $z \in U$, $z \neq 0$

2) For every $\delta > 0$ there exists $z \in U$ with $\|z\| < \delta$ and $V(z) > 0$

3) $V(0) = 0$.

Then the origin 0 is an unstable equilibrium.

Remark. The Theorem Th. 5.7, p. 174 formulated in the course book is a bit stronger. It has the same conclusion with the condition 1) changed to a weaker one:

1) $V_f(z) = \nabla V \cdot f(z) > 0$ for every point $z \in U$, where $V(z) > 0$, and 3) is not required.

Proof of the weaker variant of the Theorem

The idea of the proof is to show that any trajectory starting from a point ξ arbitrarily close to 0 where $V(\xi) > 0$ will leave a fixed ball $B(R,0)$ such that $\overline{B(R,0)} \subset U$.

We point out that for any part of the trajectory $\varphi(t,\xi)$ of the maximal solution in U the function $V(\varphi(t,\xi))$ is monotone increasing because $\frac{d}{dt}V(\varphi(t,\xi)) = V_f(\varphi(t,\xi)) > 0$.

It means that $\varphi(t,\xi)$ stays on a positive distance from the origin because $V_f(z)$ is continuous and $V_f(0) = 0$. It in turn means that $(\nabla V \cdot f)(\varphi(t,\xi)) = \frac{d}{dt}V(\varphi(t,\xi) \geq K > 0$ for all $t \in I_\xi \cap \mathbb{R}^+$.

To prove this inequality one can carry out a more formal argument that follows.

Let $\xi \in B(R,0)$ be an arbitrary point where $V(\xi) > 0$. V is a continuous function and $V(0) = 0$. It implies that there is $0 < \varepsilon < R/2$ such that $V(z) < V(\xi)/2$ for $\|z\| < \varepsilon$.

Therefore the trajectory $\varphi(t,\xi)$ must stay outside the ball $B(\varepsilon, 0)$ for all $t \in I_\xi \cap R^+$.

The function $(\nabla V \cdot f)(z)$ is continuous in U and must attaint its minumum $K = \min_{z\in\overline{B(R,0)}\setminus B(\varepsilon,0)} (\nabla V \cdot f)(z)$ on the compact set $\overline{B(R,0)} \setminus B(\varepsilon, 0)$ that is a slab between two spheres. The number K is positive $K > 0$ because $(\nabla V \cdot f)(z) > 0$ for $z \in U$ outside the origin.

Therefore

$$(\nabla V \cdot f)(\varphi(t,\xi)) = \frac{d}{dt}V(\varphi(t,\xi)) \geq K > 0, \quad \forall t \in I_\xi \cap \mathbb{R}^+$$

and by the integration of the left and right hand side over $[0,t]$ we get

$$V(\varphi(t,\xi) \geq Kt + V(\xi), \quad \forall t \in I_\xi \cap \mathbb{R}^+$$

There are two possibilities depending on if $I_\xi \cap \mathbb{R}^+$ is a bounded interval or $\mathbb{R}^+ \subset I_\xi$. In the first case the trajectory $\varphi(t,\xi)$ must leave any compact in G in particular the ball $\overline{B(R,0)}$. In the second case having possibility to take t arbitrary large in the inequality $V(\varphi(t,\xi) \geq Kt + V(\xi)$ leads to conclusion, that for some time $t_* > 0$ large enough $V(\varphi(t_*,\xi))$ will become larger than $\max_{z\in\overline{B(R,0)}} V(z)$ - the maximum of $V(z)$ over the half ball $\overline{B(R,0)}$. It means that the point $\varphi(t_*,\xi)$ of the trajectory must be outside the ball $\overline{B(R,0)}$ at such time t_*.

Therefore according to the definition, the origin 0 is an unstable equilibrium, because there are trajectories starting arbitrarily close to the equilibrium 0, such that they move outside the ball $\overline{B(R,0)} \subset G$ at some time t_*.■

Remark. If we suppose in the formulation of the theorem above that $V_f(z) > 0$ for all $z \in U$, $z \neq 0$, then the origin is a repeller, meaning that for some ball $B(R,0)$ around the origin, any solution $x(t) = \varphi(t,\xi)$ with $\xi \in B(R,0)$ will leave this ball in finite time.

Example on instability 1

Consider the system

$$\begin{aligned} x' &= x^3 + yx^2 \\ y' &= -y + x^3 \end{aligned}$$

Show that the origin is unstable equilibrium by using the test function $V(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$.

Point out that the linearization has matrix $J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ that is degenerate (determinant is zero, meaning that in this case there is at least one zero eigenvalue).

Therefore the Grobman - Hartman theorem cannot be applied.

$$\begin{aligned} V_f(x,y) &= \begin{bmatrix} x \\ -y \end{bmatrix} \cdot \begin{bmatrix} x^3 + yx^2 \\ -y + x^3 \end{bmatrix} = \\ y^2 + x^4 - yx^3 + x^3y &= y^2 + x^4 > 0, \quad (x,y) \neq (0,0) \end{aligned}$$

$V(x,y) > 0$ on the x-axis outside the origin, including points arbitrarily close to the origin. In fact $V(x,y) > 0$ even in the cone $|y| < |x|$.

There is a ball $B(0,R)$ around the origin such that trajectories starting on the x-axis arbitrary close to the origin will leave it in finite time by the Lyapunov instability theorem. Therefore the origin is unstable.

Example on instability 2

Consider the following system of ODEs. Prove the instability of the equilibrium point in the origin, of the following system

$$\begin{cases} x' = & x^5 + y^3 \\ y' = & x^3 - y^5 \end{cases} \qquad \textbf{(4p)}$$

using the test function $V(x, y) = x^4 - y^4$ and Lyapunov's instability theorem.

Solution.

Denoting $f(x, y) = \begin{bmatrix} x^5 + y^3 \\ x^3 - y^5 \end{bmatrix}$, consider how the function $V(x, y) = x^4 - y^4$ changes along trajectories of the system.

$$\begin{aligned} V_x(x, y) &= f(x, y) \cdot \nabla V(x, y) = \begin{bmatrix} x^5 + y^3 \\ x^3 - y^5 \end{bmatrix} \cdot \begin{bmatrix} 4x^3 \\ -4y^3 \end{bmatrix} = \\ &= x^5 4x^3 + y^3 4x^3 - x^3 4y^3 + y^5 4y^3 = x^5 4x^3 + y^5 4y^3 = 4(x^8 + y^8) > 0. \end{aligned}$$

Point out that the function $V(x, y) = x^4 - y^4$ is positive in the cone $|y| < |x|$ or simply along the line $y = x/2$, including points arbitrarily close to the origin. It implies according to the instability theorem, that the origin is an unstable equilibrium.■