Main ideas and tools in the course on ODE

1. Integral form of I.V.P. to ODEs
2. Grönwall's inequality for showing uniqueness and continuity with respect to data.
3. Transition mapping. Orbits of solutions, phase portrait.
4. Generalised eigenspaces of matrices. Basis of generalized eigenvectors.
5. Jordan form of matrices. Functions of matrices, in particular exponent and logarithm.
6. Transition matrix. Chapmen-Kolmogorov relations.
7. Monodromy matrix. Floquet theory for periodic linear systems.
8. Stability and instability of equilibrium points.
9. Linearization and Grobman - Hartman theorem. (iff $\mathrm{Re}(\lambda) \neq 0$)
10. Lyapunov functions (for stability, instability, and for finding positively invariant sets).
11. ω - limit sets. LaSalle's invariance principle for hunting ω - limit sets "living" in $V_f^{-1}(0)$.
12. Idea of solving integral equations by iterations (Banach's contraction priniple).

Examples of typical problems

Example on an application of Jordan matrix

For one particular solution of the system $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$ with a real matrix A, the first component has the form $x_1 = t^2 + t\sin(t)$.

1. Which smallest size can the real matrix A have? **(4p)**

   Solution.

   The term $t\sin(t)$ in the solution is a sign that the Jordan form of the matrix A has a Jordan block corresponding to the eigenvalue $\lambda_1 = i$ that has multiplicity at least 2, for example $\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$ or multiplicity 3 : $\begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}$ etc. On the other hand te matrix A is real and therefore it's characteristic plolynomial has real coefficients and therefore all complex eigenvalues must appear as conjugate pairs: the matrix A must have the eigenvalue $\lambda_2 = -i$ havingthe same multiplicity as λ_1, at least 2 and with corresponding Jordan block $\begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}$.

The presence of the term t^2 in one component of a solution shows that the matrix A must have the eigenvalue $\lambda = 0$ with multiplicity at least 3 with correspoding Jordan block $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

All these observations imply that the real matrix A must have dimensions at least 7×7, because the sum of dimensions of sizes of Jordan blocks is at least $2+2+3=7$.■

Example of transition mapping.

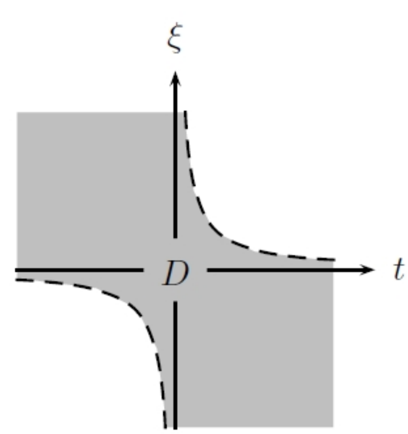
Example 4.33 of a transition map.
$G = \mathbb{R}$; $f : G \to \mathbb{R}$; $f(x) = x^2$; for $\xi = 0$; $x(t) \equiv 0$.
Initial data $x(0) = \xi$

$$\begin{aligned}
\frac{dx}{dt} &= x^2; \quad \int \frac{dx}{x^2} = \int dt; \\
-\frac{1}{x} &= t + C \\
-\frac{1}{x} &= t - \frac{1}{\xi}; \quad -\frac{1}{x} = \frac{t\xi - 1}{\xi} \\
x(t) &= \frac{\xi}{(1 - t\xi)}
\end{aligned}$$

The maximal interval for $\xi = 0$; $x(t) \equiv 0$. is $I_\xi = \mathbb{R}$
The maximal interval for $\xi > 0$, $I_\xi = (-\infty, 1/\xi)$.
The maximal interval for $\xi < 0$, $I_\xi = (1/\xi, \infty)$

$$\varphi(t,\xi) = \frac{\xi}{(1 - t\xi)}; \quad D(\varphi) = \{(t,\xi) \in \mathbb{R} \times \mathbb{R}; \quad t\xi < 1\}$$

The domain D of φ is an open set. The function $\varphi(t,\xi)$ is continuous and even locally Lipschitz.



Example of a transition mapping and maximal solutions (a bit more complicated).

1) Solve the initial value problem

$$\dot{x} = t\,x^3, \qquad x\,(1) = \xi$$

and find maximal intervals for solutions. Give a sketch of the domain for the transfer mapping $\varphi(t,1,\xi)$=$x(t)$ in the (t,x) plane.

2) Can one conclude which maximal interval have solutions to the similar equation

$$\dot{x} = t^3 x$$

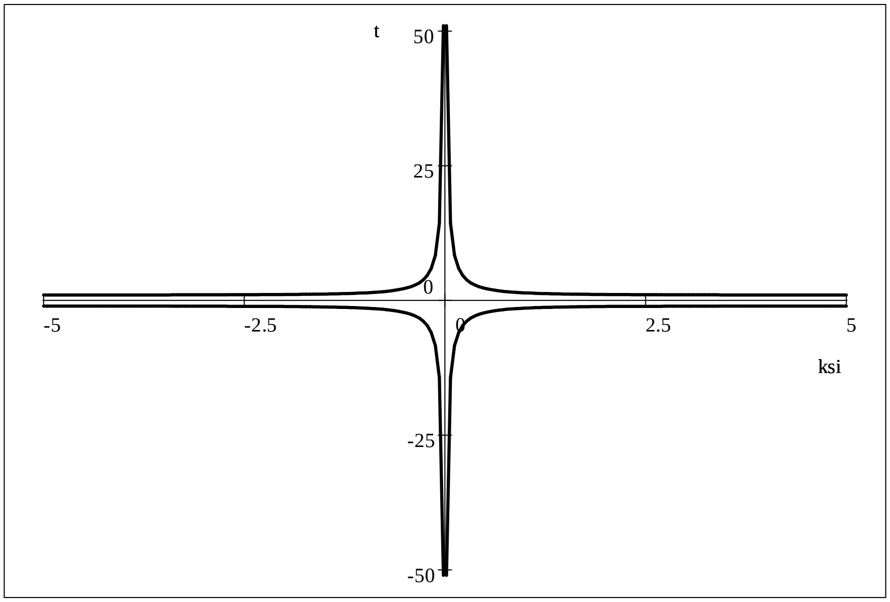
without solving it?

Solution.

1) It is the equation with separable variables.

$$\begin{aligned}
\frac{dx}{dt} &= tx^3; \qquad x\,(1) = \xi \\
\int \frac{dx}{x^3} &= \int tdt \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - C \\
C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2\xi^2} = \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{\xi^2 t^2}{2\xi^2} - \frac{1+\xi^2}{2\xi^2} = \frac{\xi^2 t^2 - \left(1+\xi^2\right)}{2\xi^2} \\
x^2 &= \frac{\xi^2}{\left(1+\xi^2\right) - \xi^2 t^2} = \frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2} \\
x &= \sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi > 0 \\
x &= -\sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi < 0 \\
x &\equiv 0, \quad \xi = 0, \quad -\text{equilibrium}, \quad t \in \mathbb{R} \\
\left(1+\xi^2\right)/\left(\xi^2\right) &> t^2; \quad t \in \left(-\sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}, \sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}\right) OPEN!!!
\end{aligned}$$

The maximal intervals for these solutions are open in accordance with the general theory. One solution $x \equiv 0$ is defined on the whole $\mathbb{R}$. We draw boundaries of the domain for $\varphi(t,1,\xi)$.



Example of an equation with "eternal" solutions.

The equation $\dot{x} = t^3 x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$, $R > 0$ the estimate $|t^3 x| \leq R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies that the maximal existence interval for all solutions to this equation is $\mathbb{R}$.

Estimating Lyapunov functions V and their derivatives $V_f = \nabla V \cdot f$ along trajectories.

Investigation of the sign of functions V and $V_f = \nabla V \cdot f$.

Choosing a Lyapunov's function for stability analysis: it must be positive definite: $V(0) = 0$, $V(x) > 0$, $x \neq 0$.

This property lets to use some of the level sets also as boundaries for 1) positively invariant sets and 2) regions (or domains) of attraction for asymptotycally stable equilibrium points.

(For instability analysis it is enough to find a test function V such tat it is positive arbitrarily close to the equilibrium point in the origin, for example on a line through the origin or in a cone with the vertex in the origin).

The second step in finding Lyapunovs functions is consideration of the sign of the function $V_f(x) = \nabla V \cdot f(x)$. This finction gives the rate of change for $V(x)$ trajectories $x(t)$ of the differential equation $x' = f(x)$ without solving the equation, because $\frac{d}{dt}V(x(t)) = \nabla V \cdot f(x(t))$.

The choice of test functions

1. The simplest choice of a test function V for using in Lyapunovs theorems is $V(x,y) = x^2 + y^2$ having level sets being circles around the origin. It is often our first choice. Sometimes test functions like $V(x,y) = ax^2 + bxy + cy^2$ with indefinite terms xy can be convenient if they are positive definite.

2. Test functions as a sum of kinetic and potential energy.One dimensional Newton equation. First integrals

For systems in the form

$$\begin{aligned} x' &= y, \\ y' &= -ay - g(x) \end{aligned}$$

defined for all $(x,y) \in \mathbb{R}^2$ equivalent to the Newton equation

$$x'' = -ax' - g(x),$$

with potential force $-g(x)$ it is natural and optimal to choose a test function as a sum of the kinetic energy $\frac{1}{2}y^2$ and $G(x) = \int_0^x g(s)ds$:

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)ds$$

If the force is an odd function such that $xg(x) > 0$, $x \neq 0$, and $g(0) = 0$ this test function $V(x,y)$ will be positive definite in some region around the origin.

The derivative V_f of V along trajectories for the friction force equal to $-ay$, $a > 0$

$$
\begin{aligned}
(\nabla V \cdot f)(x,y) &= \left(\frac{\partial}{\partial x}V\right) f_1 + \left(\frac{\partial}{\partial y}V\right) f_2 \\
&= g(x)y + x\left(-g(x)\right) - ay^2 = -ay^2 \leq 0
\end{aligned}
$$

The Lyapunov stability theorem would imply that the origin is a stable equilibrium point. Depending on how the potential $G(x) = \int_0^x g(s)ds$ behaves and on the position of other equilibrium points (zeroes of the function $g(x)$), one can find a region of attraction bounded by a level set of V that includes only one equilibrium point.

One can use the same idea in the case when the friction force in the equation above has the form: $-a\phi(y)$ with $\phi(y)y > 0$,

3. Test functions as a higher order polynomial arbitrary even powers and with arbitrary coefficients.

A flexible choice of a test function $V(x,y)$ can be

$$V(x,y) = ax^m + by^n$$

with arbitrarty exponents m, n and arbitrary coefficients $a, b > 0$ that are chosen after the calculation of $V_f(x,y)$ so that $V_f(x,y) \leq 0$ or $V_f(x,y) \leq 0$ for $(x,y) \neq (0,0)$.

Example: $V(x,y) = x^2 + xy + 2y^2$. Level sets of such a test function will be ellipses with the axis rotated with respect to the coordinate system. The Cauchy inequality

$$|xy| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

helps to show that this test function is positie definite. Another way to show is to analyse $V(x,y)$ as a quadratic form.

A more general Young inequality

$$|ab| \leq \frac{a^p}{p} + \frac{b^q}{q}; \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$$

can be useful for investigating polynomials of higher degree in f:

This property $V(x) > 0,\ x \neq 0,\ V(0) = 0$ is a condition in the theorem by Lyapunov on stability. It implies i particular that level sets of V close to the origin are closed curves.

Analysis of V_f

We like to have $V_f = \nabla V \cdot f(x)$ negative definite $V_f(x) < 0$ or at least $\nabla V \cdot f(x) \leq 0$ for $x \neq 0$.

Here f is the right hand side ("velocity") in the differential equation of interest: $x' = f(x)$. It makes $\frac{d}{dt}(x(t)) = \nabla V \cdot f(x(t))-$ showing how the test function V changes along trajectories $x(t)$.

Analysis of $V_f^{-1}(0)$

After calculating $V_f(x)$ we check the set $V_f^{-1}(0)$ where $V(x) = 0$. Why it is interesting?

The La Salle's invariance principle states that all ω - limit sets of trajectories $x(t)$ inside the domain where $\nabla V \cdot f(x) \leq 0$ is valid belong to the set $V_f^{-1}(0)$ and they belong even to a smaller part of it that is the maximal invariant subset in $V_f^{-1}(0)$.

How to apply La Salle's invariance principle ?

i) The set $V_f^{-1}(0)$ is easy to identify, as a set of zeroes to V_f (in plane in most of our examples). It is usually one or both coordinate axes.

ii) The maximal invariant set inside $V_f^{-1}(0)$ (in the plane it will be a set of curves) is easy to check invariant sets just by looking on velocities (values of $f(x, y)$) on the set $V_f^{-1}(0)$ and checking if they go along curves forming $V_f^{-1}(0)$ or they go across.

It implies in particular that if in addition to the inequality $\nabla V \cdot f(x) \leq 0$ the set $V_f^{-1}(0)$ includes only an invariant set consisting of the origin, then, the origin is asymptotically stable equilibrium.

Example.

Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

Hint: applying Lyapunovs theorem, use the elementary inequality

$$|xy| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

to estimate possible indefinite terms with xy in the expression for $V_f(x, y)$.

Solution. Choose a test function $V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$

$V_f = \nabla V \cdot f = x(-x - 2y + xy^2) + y\left(3x - 3y + y^3\right) = xy - x^2 - 3y^2 + y^4 + x^2y^2$

$= -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + \underbrace{xy}_{indefinite\_term!} \leq -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + 0.5x^2 + 0.5y^2$

We apply the inequality $2xy \leq \left(x^2 + y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

$V_f \leq -x^2\left(0.5 - y^2\right) - y^2\left(2.5 - y^2\right)$

It implies that $V_f < 0$ for $(x, y) \neq (0, 0)$ and $|y| < 1/\sqrt{2}$.Therefore the Lyapunof function V is "strong" and therefore the origin is asymptotically stable.

The region of attraction is bounded by the largest level set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely the circle: $\left(x^2 + y^2\right) < 1/2$.

The second idea for choosing Lyapunov functions is choice of V of **polynomilas with arbitrary even powers and arbitrary coefficients.**

Another more clever choice of a test function as

$$V(x, y) = ax^m + by^n$$

in particular $V(x, y) = 3x^2 + 2y^2$ works in this particular case:

$V_f = 6x(-x - 2y + xy^2) + 4y(3x - 3y + y^3) = 4y^4 - 12y^2 - 6x^2 + 6x^2y^2 = -4y^2\left(3 - y^2\right) - 6x^2\left(1 - y^2\right) < 0$

for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ that fits into the stripe $|y| < 1$ in the plane is a region of attraction for the asymptotically stable equilibrium in the origin.

One can also observe the asymptotic stability of the origin here by linearization with variational matrix

$A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2 + 4\lambda + 9 = 0$, and calculating eigenvalues: $-i\sqrt{5} - 2, i\sqrt{5} - 2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the domain of attraction.

Problem on stability of equilibrium points and on domains of attraction.

Consider the following system of ODEs. $\begin{cases} x' = & 1 - xy \\ y' = & x - y^3 \end{cases}$

Find all equilibrium points and investigate their stability. Find domains of attraction for possible asymptotically stable equilibrium points. **(4p)**

Solution.

Equilibrium points are $(1,1)$ and $(-1,-1)$ can be found by substitution. $x = y^3$, $1 = xy = y^4$.

Jacoby matrix of the right hand side is $J(x,y) = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix}$; $J(1,1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$; $J(-1,-1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. $\det(J(1,1)) = 4$, $\operatorname{tr}(J(1,1)) = -4$. Therefore the equilibrium point $(1,1)$ is asymptotically stable.

$\det(J(-1,-1)) = -4$. Therefore the linearized around $(-1,-1)$ system has a saddle point and the equilibrium point $(-1,-1)$ is unstable.

We shift the origin of the coordinate system into the point $(1,1)$ by introducing new variables $u = x - 1, v = y - 1$ and $x = u + 1$, $y = v + 1$.

$$\begin{cases} u' = & -u - v - uv \\ v' = & u - 3v - 3v^2 - v^3 \end{cases}$$

Consider a test function $E(u,v) = \frac{1}{2}\left(u^2 + v^2\right)$

$$\begin{aligned} \frac{d}{dt}E(u(t),v(t)) &= \begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} -u - v - uv \\ u - 3v - 3v^2 - v^3 \end{bmatrix} = \\ &= -u^2 - uv - u^2 v + uv - 3v^2 - 3v^3 - v^4 = \\ &= -u^2(1 - v) - 3v^2 \underbrace{\left(1 + v + v^2\right)}_{>0} < 0 \\ if \quad v &< 1, \quad (u,v) \neq (0,0) \end{aligned}$$

The largest circle in (u,v) plane satisfying the condition $v \leq 1$ has radius 1. Therefore the circle of radius 1 around the equilibrium point $(1,1)$ is the domain of attraction for the asymptotically stable equilibrium $(1,1)$ of the original system of ODEs.$\blacksquare$

Application of Poincare - Bendixson theorem

The generalized Poincare-Bendixson's theorem gives a complete description of possible types of ω - limit sets in the plane $\mathbb{R}^2$.

Theorem (generalized Poincare-Bendixson)

Let M be an open subset of $\mathbb{R}^2$ and $f : M \to \mathbb{R}^2$ and $f \in C^1$. Fix $\xi \in M$ and suppose that the closure of $\Omega(\xi) \neq \emptyset$, is compact, connected and contains only finitely many equilibrium points.

(i) $\Omega(\xi)$ is an equilibrium point

(ii) $\Omega(\xi)$ is a periodic orbit

(iii) $\Omega(\xi)$ consists of finitely many fixed points $\{x_j\}$and non-closed orbits γ such that ω and α - limit points of γ belong to $\{x_j\}$.

In practice the only reliable way of applying the Poincare-Bendixson theorem is to find a compact positively invariant set $K \subset M$ such that $\xi \in K$.

Then according to the Main theorem about ω - limit sets any solution with orbit O_+ in the compact K will have a non-empty compact ω - limit set $\Omega(\xi)$ in K.

If in addition there are no equilibrium points in K, then the Poincare-Bendixson theorem implies that $\Omega(\xi)$ is an orbit of a perioduic solution.

Finding a positive invariant set for using Poincare - Bendixson theorem and testing the absence of equilibrium points in a positive invariant set.

We try to find an ring shaped compact set K that is positively invariant and need to check three conditions:

i) The outer boundary of the ring (using a level set of a test function, or a polygon shaped domain testing velosities on each segment of it's boundary)

ii) The inner boundary of the ring (using a level set of a test function, or linearization for identifying a repeller inside a large postively invariant set by applying the Grobman - Hartman theorem)

iii) Show that the found compact positively invariant ring shaped set includes no equilibrium points. (this condition is often missed by students)

Then we can conclude that any solution starting with $\xi \in K$, will have the orbit $O_+(\xi)$ in the compact K and will have a non-empty compact ω - limit set $\Omega(\xi)$ in K. This ω - limit set must be a periodic orbit according to Poincare - Bendixson theorem.

Example.

Consider the following system of ODEs. $\left\{ \begin{array}{ll} x' = & y \\ y' = & -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{array} \right.$.

Show that this system has a non-trivial periodic solution. **(4p)**

Point out that the origin is outside the domain of the equation.

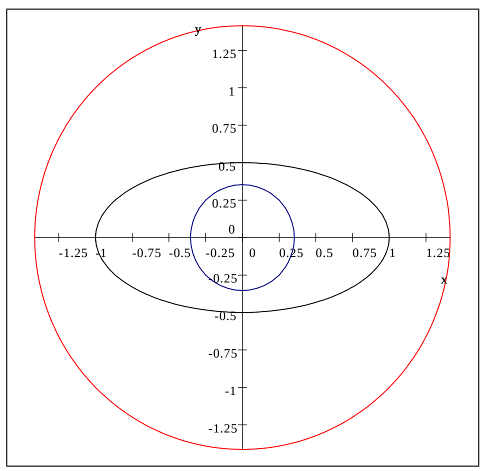
Solution.

Consider the test function $E(x,y) = \frac{1}{2}\left(x^2+y^2\right)$

$\frac{d}{dt}E(x(t),y(t)) = E_f(x,y) = \nabla E \cdot f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} y \\ -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{bmatrix}$

$= -y^2\left[\ln\left(x^2+4y^2\right)\right] \left\{ \begin{array}{ll} \geq 0 & 0 < x^2+4y^2 < 1 \\ \leq 0 & \quad x^2+4y^2 > 1 \end{array} \right.$

The boundary curve separating domains with different signs of $x^2+4y^2 = 1$



is the ellipse with halv axes 1 and 1/2 i the x - direction with center in the origin. Therefore any circle with the center in the origin inside this ellipse is never entered by a trajectory. Similarly any circle with the center in the origin outside this ellipse is never left by a trajectory. Such two circles build an annulus that is a compact positively invariant set for this system of ODEs.

For example an annulus $1/4 \leq x^2+y^2 \leq 1$ satisfies this conditions. This annulus contains no equilibrium points, because the origin is the only equilibrium point. The compact positively invariant set R must include at least one $\omega-$ limit set $\Omega(\xi)$. R does not include any equilibrium point and according to Poincare-Bendixson theorem this $\omega-$ limit set $\Omega(\xi)$ must be the orbit of a periodic solution.■

1.

Example. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x,y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vectorfunction $F(x,y)$ we conclude that

$V_f = \nabla V \cdot f = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - \left(2x^2 + y^2\right)\right)\left(2x^2 + y^2\right)$.

It implies that the elliptic shaped ring: $R = \{(x,y) : 0.5 \leq (2x^2 + y) \leq 2\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$.

The origin is the only equilibrium point of the system**. It is not so easy to see from the system of equations itself.** But one can see it easier by cheching first zeroes of $V_f(x,y)$ that is a scalar function and evidently must be zero in all equilibrium points..

We observe that $V(x,y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot f(x,y) = 0$ only if $(x,y) = (0,0)$ or if $(2x^2 + y^2) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x,y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x,y) = (0,0)$. The equation for equilibrium points on the level set $(2x^2 + y^2) = 1$ is the following:

$$\begin{cases} 0 = x - 2y - x = -2y \\ 0 = 4x + y - y = 4x \end{cases}$$

The compact positively invariant set R must include at least one $\omega-$ limit set $\Omega(\xi)$ in R. R does not include any equilibrium point and according to Poincare-Bendixson theorem this $\omega-$ limit set $\Omega(\xi)$ must be the orbit of a periodic solution.$\blacksquare$

Example

Show that the following system of ODE-s has a periodic solution.

$$\begin{cases} x' = 4x + y - x\left(5x^2 - 2\sqrt{3}xy + 7y^2\right) \\ y' = -x + 4y - y\left(5x^2 - 2\sqrt{3}xy + 7y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Hint. The Cauchy inequality $|ab| \leq 0.5\left(a^2 + b^2\right)$ can be useful for analysis here.

Solution. We like to apply the Poincare-Bendixson theorem to prove that the system has a periodic solution by showing that some of it's trajectories must have a periodic orbit as an ω-limit set. To show it we find a positively - invariant set that does not include equilibrium points. By the Poincare-Bendixson theorem all trajectories starting in this positively invariant set will have an ω limit set that is a periodic orbit.

We consider the test function $V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$ and try to find two such circles (level sets to $V(x,y)$) that they bound a positively invariant set .

$$V_f(x,y) = \begin{bmatrix} 4x + y - x\left(5x^2 - 2\sqrt{3}xy + 7y^2\right) \\ -x + 4y - y\left(5x^2 - 2\sqrt{3}xy + 7y^2\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$x\left(4x + y - x\left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right) + y\left(-x + 4y - y\left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right) =$$

$$\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right]\left(x^2 + y^2\right).$$

We see that $V_f(x,y) < 0$ for $4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right) < 0$ and $V_f(x,y) > 0$ for $4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right) > 0$.

The curve $4 = \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)$ is an ellipse (red curve on the picture) because the expression $\left(5x^2 - 2\sqrt{3}xy + 7y^2\right)$ is positive definite by the Cauchy inequality $|xy| \leq 0.5\left(x^2 + y^2\right)$:

$5x^2 - 2\sqrt{3}xy + 7y^2 \geq 5x^2 - 2\sqrt{3}\left(x^2 + y^2\right)0.5 + 7y^2 = x^2\left(5 - \sqrt{3}\right) + y^2(7 - \sqrt{3}) > 0$, $(x,y) \neq 0$.

One can also observe it by investigating eigenvalues of the matrics corresponding this quadratic form:

$Q(x,y) = 5x^2 - 2\sqrt{3}xy + 7y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. $\begin{bmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{bmatrix}$. The matrix $\begin{bmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{bmatrix}$ has eigenvalues: $4, 8 > 0$, eigenvectors are orthogonal vectors $\begin{bmatrix} \sqrt{3}/2 \\ 0.5 \end{bmatrix}$ and $\begin{bmatrix} -0.5 \\ \sqrt{3}/2 \end{bmatrix}$ that define the orientation of the ellips.

This ellipse separates the area where $V_f(x,y) < 0$ and trajectories of the system go inside circles, that are level sets of $V(x,y)$ from the area where $V_f(x,y) > 0$ and trajectories of the system go outside circles that are level sets of $V(x,y)$.

Finding two circles $x^2 + y^2 = R^2$ and $x^2 + y^2 = r^2$, $R > r > 0$ such that the first one is completely outside the ellipse $4 = \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)$ and the second one is completely inside the ellipse, will give us the desired ring shaped positively invariant set: $r^2 < x^2 + y^2 < R^2$. It is intuitively evident that such R - large enough and r - small enough exist.

Then we must check that the ring shaped positively invariant set does not contain any equilibrium points. In any equilibrium point we must have $V_f(x,y) = 0$. It implies that $\left(x^2 + y^2\right)\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right] = 0$ that gives us that an equilibrium point must be in the origin, that is outside our positively invariant set, or on the ellipse $4 =$

$\left(5x^2 - 2\sqrt{3}xy + 7y^2\right)$. We observe from the ODE, that on this ellipse $x' = y$ and $y' = -x$. Therefore equilibrium points can be only the origin $(x, y) = (0, 0)$. It is outside the ellipse and outside the positively invariant set.

Therefore all trajectories starting in the positively invariant set $r^2 < x^2 + y^2 < R^2$ must have an ω - limit set inside it and this limit set must be a periodic orbit by the Poincare-Bendixson theorem. Therefore the system must have at least one periodic orbit inside the positively invariant set.

We can also find some explicit estimates for R and r.

We consider the expression $\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right]$ and try to find a circle $x^2+y^2 = R^2$ such that $\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right] < 0$ on it.

$\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right] \le \left[4 - 5x^2 + 2\sqrt{3}\,|xy| - 7y^2\right] \le \left[4 - 5x^2 + \sqrt{3}\,(x^2 + y^2) - 7y^2\right]$

$\left[4 - 5x^2 + \sqrt{3}\,(x^2 + y^2) - 7y^2\right] = 4 - \left(5 - \sqrt{3}\right)x^2 - \left(7 - \sqrt{3}\right)y^2 \le 4 - \left(5 - \sqrt{3}\right)x^2 - \left(5 - \sqrt{3}\right)y^2 \le 0.$

Therefore for $x^2 + y^2 = R^2 \ge 4/\left(5 - \sqrt{3}\right)$ the desired inequality $V_f(x, y) \le 0$ is valid. We found an outer boundary of the ring shaped positively invariant set.

$R \ge 2$ for example would work.

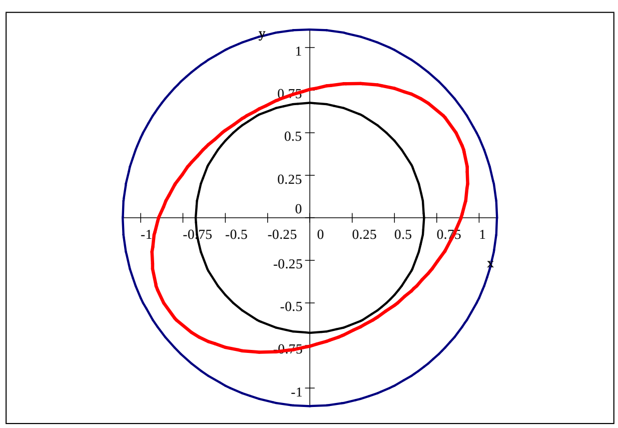
We consider the expression $\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right]$ and try to find a circle $x^2+y^2 = r^2$ such that $\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right] \ge 0$ on this circle.

$\left[4 - \left(5x^2 - 2\sqrt{3}xy + 7y^2\right)\right] \ge \left[4 - \left(5x^2 + 2\sqrt{3}\,|xy| + 7y^2\right)\right] \ge$

$\left[4 - \left(5x^2 + 2\sqrt{3}\,|xy| + 7y^2\right)\right] \ge \left[4 - \left(5 + \sqrt{3}\right)x^2 - \left(7 + \sqrt{3}\right)y^2\right] \ge \left[4 - \left(7 + \sqrt{3}\right)x^2 - \left(7 + \sqrt{3}\right.\right.$

0

Therefore for $x^2 + y^2 = r^2 < 4/\left(7 + \sqrt{3}\right)$ the desired inequality $V_f(x, y) \ge 0$ is valid. We have found the internal boundary for the ring shaped positively invariant set that finally is defined by $\left\{4/\left(7 + \sqrt{3}\right) < x^2 + y^2 < 4/\left(5 - \sqrt{3}\right)\right\}$. Check the picture of the ellips and two circles that we found.



Problems on Floquet theory and linear non-autonomous equations

5. Exercise 2.21. p.58.

Consider the Hill equation $y'' + a(t)y = 0$; $a(t+p) = a(t)$.with periodic $a(t)$ with period $p = 1$. The vector form with $x_1(t) = y(t)$, $x_2(t) = y'(t)$ of the equation is:

$$\begin{aligned} x' &= A(t)x \\ A(t) &= \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix} \end{aligned}$$

We choose $a(t)$ as a piecewise constat periodic function:

$$a(t) = \begin{cases} \omega^2, & m \le t < m + \tau \\ 0, & m + \tau \le t < m + 1 \end{cases}$$

Here $\tau \in (0,1)$, $\omega = \pi/\tau$.

Consider the transfer matrix solution $\Phi(t,0)$ and show that its first column $\Phi_1(t,0)$ is periodic with period 2, and it's second column $\Phi_2(t,0)$ is unbounded with it's first element at times $t = n$ equal to $(-1)^n n(1-\tau)$.

Solution. The monodromy matrix has the followinf structure:

$$\Phi(1,0) = \Phi(1,\tau)\Phi(\tau,0) = \exp((1-\tau)A_2)\exp(\tau A_1)$$

where according to the definition of $A(t)$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} = A(t)\ ,\ t \in (0,\tau)$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A(t), \quad t \in (\tau, 1)$$

Eigenvectors to A_1 are: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left\{ \begin{bmatrix} -\frac{i}{\omega} \\ 1 \end{bmatrix} \right\} \leftrightarrow i\omega$,
$\left\{ \begin{bmatrix} \frac{i}{\omega} \\ 1 \end{bmatrix} \right\} \leftrightarrow -i\omega$.

Check the first of eigenvectors:
$\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\omega \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{aligned} v_2 &= i\omega v_1 \\ -\omega^2 v_1 &= i\omega v_2 \end{aligned}$$

$$x_*(t) = \left(\begin{bmatrix} -\frac{i}{\omega} \\ 1 \end{bmatrix} \exp(i\omega t) \right) = \begin{bmatrix} -\frac{i}{\omega} \\ 1 \end{bmatrix} (\cos(\omega t) + i\sin(\omega t)) = \begin{bmatrix} -\frac{i}{\omega}(\cos t\omega + i\sin t\omega) \\ \cos t\omega + i \sin t\omega \end{bmatrix}$$

; $\operatorname{Re} x_*(t) = \begin{bmatrix} \frac{1}{\omega}(\sin t\omega) \\ \cos t\omega \end{bmatrix}$; $\quad \operatorname{Im} x_*(t) = \begin{bmatrix} -\frac{1}{\omega}\cos t\omega \\ \sin t\omega \end{bmatrix}$

We like to build using these two linearly independent solutions, one solution with initial data $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and one solution with initial data $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is easy to see that the following solutions satisfy these initial conditions and can be collected into the transfer matrix:

$$\begin{aligned} \Phi(t,0) &= [-\omega \operatorname{Im} x_*(t), \operatorname{Re} x_*(t)] = \begin{bmatrix} \cos t\omega & \frac{1}{\omega}(\sin t\omega) \\ -\omega \sin t\omega & \cos t\omega \end{bmatrix} \\ \Phi(\tau,0) &= \begin{bmatrix} \cos \tau\omega & \frac{1}{\omega}(\sin \tau\omega) \\ -\omega \sin \tau\omega & \cos \tau\omega \end{bmatrix} \end{aligned}$$

We will calculate $\Phi(t,\tau)$ for $t \in (\tau, 1]$.

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A_2 is a Jordan block with eigenalue $\lambda = 0$.

Then $\Phi(t,\tau) = \exp\left((t-\tau)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = e^{0(t-\tau)}\begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}$ according to formulas for a Jordan block.

Then $\Phi(1,\tau) = \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix}$;

The monodromy matrix is calculated as:

$$\begin{aligned} \Phi(1,0) &= \Phi(1,\tau)\Phi(\tau,0) = \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos \tau\omega & \frac{1}{\omega}(\sin \tau\omega) \\ -\omega \sin \tau\omega & \cos \tau\omega \end{bmatrix} \\ &= \begin{bmatrix} \cos \tau\omega - \omega(\sin \tau\omega)(1-\tau) & \frac{1}{\omega}\sin \tau\omega + (\cos \tau\omega)(1-\tau) \\ -\omega \sin \tau\omega & \cos \tau\omega \end{bmatrix} \end{aligned}$$

If $\omega = \pi/\tau$, then the monodromy matrix is

$$\begin{aligned} \Phi(1,0) &= \begin{bmatrix} \cos \pi - \omega(\sin \pi)(1-\tau) & \frac{1}{\omega}\sin \pi + (\cos \pi)(1-\tau) \\ -\omega \sin \pi & \cos \pi \end{bmatrix} \\ &= \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Eigenvalues of this triangular monodromy matrix are both equal to $\lambda_{1,2} = -1$.

Checking the matrix $\Phi(1,0) - (-1)I = \begin{bmatrix} 0 & -(1-\tau) \\ 0 & 0 \end{bmatrix}$ we find only one linearly independent eigenvector to $\Phi(1,0)$ is $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

1) Therefore there must exist unbounded solutions because the multiple $\lambda_{1,2} = -1$ is not semisimple. (!!!!)

2) Therefore $(\lambda_{1,2})^2 = 1$. It implies by a Corollary previous time that the solution with initial data equal to the corresponding eigenvector e_1 has the period $2p = 2$ that is the double period of the system. In this particular case the period of coefficients is $p = 1$.

$$\begin{aligned} A_1 v &= \lambda v, \quad v \text{ - an eigenvector} \\ x_*(t) &= \exp(t\lambda) v \quad \text{is a solution to} \\ x' &= A_1 x \end{aligned}$$

This solution is the first column in $\Phi(t,0)$, because the corresponding eigenvector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - is the initial condition for the first column in $\Phi(t,0)$.

In time points $t = pn = n$ the second column in $\Phi(t,0)$ is equal to the second column in $\Phi(1,0)^n$ -that is the n - th power of the monodromy matrix that coinsides with $\Phi(t,0)$ for t equal to integer number of periods.

$\Phi(1,0)^2 = \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2\tau + 2 \\ 0 & 1 \end{bmatrix}$

$\Phi(1,0)^3 = \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix}^3 = \begin{bmatrix} -1 & 3\tau - 3 \\ 0 & -1 \end{bmatrix}$

$\Phi(1,0)^4 = \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix}^4 = \begin{bmatrix} 1 & -4\tau + 4 \\ 0 & 1 \end{bmatrix}$

We observe that $\Phi(1,0)^n = \begin{bmatrix} 1 & (-1)^n n\,(1-\tau) \\ 0 & (-1)^n \end{bmatrix}$ and the exercise is finished.

■