MVE670, Linjär algebra F/TM

Lösningar ordinarie tenta 21/10-23

1 (a) Antag $L: Q + t\mathbf{v},\ t \in \mathbb{R}$

$$\pi_1:\ x - 2y + 2z = -2 \implies \mathbf{n}_1 = (1,-2,2)$$
$$\pi_2:\ 2x - y + 2z = 2 \implies \mathbf{n}_2 = (2,-1,2)$$

Vet att $\mathbf{v} \perp \mathbf{n}_1$ & $\mathbf{v} \perp \mathbf{n}_2 \implies \mathbf{v} \parallel \mathbf{n}_1 \times \mathbf{n}_2$

$$\implies \left\{\begin{matrix}\text{längden av}\\ \mathbf{v}\ \text{irrelevant}\end{matrix}\right\} \implies \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix}1\\-2\\2\end{pmatrix} \times \begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}-2\\2\\3\end{pmatrix}$$

$$Q \in L \text{ uppfyller: } \begin{cases} x - 2y + 2z = -2 \\ 2x - y + 2z = 2 \end{cases} \overset{\substack{(\text{t.ex.})\\ z=0}}{\implies} \begin{cases} x - 2y = -2 & (-2) \\ 2x - y = 2 \end{cases}$$

$$\iff \begin{cases} x - 2y = -2 \\ 3y = 6 & \left(\tfrac{1}{3}\right) \end{cases} \iff \begin{cases} x - 2y = -2 \\ y = 2 & (2) \end{cases} \iff \begin{cases} x = 2 \\ y = 2 \end{cases}$$

$$\implies Q = (2,2,0)$$

$$\therefore\ L:\ (2,2,0) + t(-2,2,3),\ t \in \mathbb{R}$$

(b) $\pi_3 \perp \pi_1$ & $\pi_3 \perp \pi_2 \implies \mathbf{n}_3 \parallel \mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{v}$

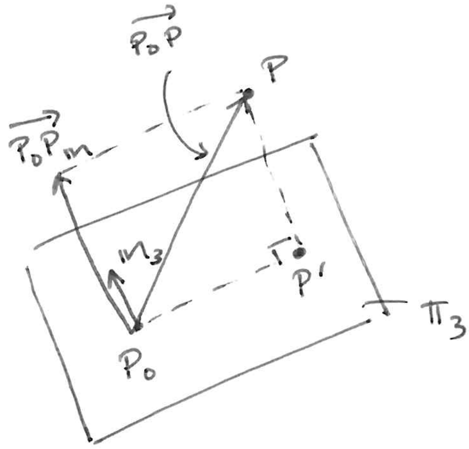
$$\implies \left\{\begin{matrix}\text{längden av}\\ \mathbf{n}_3\ \text{irrelevant}\end{matrix}\right\} \implies \mathbf{n}_3 = \mathbf{v} = (-2,2,3) \implies$$

$$\implies \pi_3:\ -2x + 2y + 3z = d$$

$P_0 \in \pi_3$ ger: $(-2)\cdot 1 + 2\cdot 2 + 3\cdot(-1) = d \iff d = -1$

$$\therefore\ \pi_3:\ -2x + 2y + 3z = -1$$

(c) $\overrightarrow{P_0P} = P - P_0 = (1,2,5)$



$$\overrightarrow{P_0P}_n = \frac{\overrightarrow{P_0P}\cdot n_3}{|n_3|^2}\, n_3 =$$

$$= \frac{(1,2,5)\cdot(-2,2,3)}{(-2)^2+2^2+3^2}(-2,2,3) =$$

$$= \frac{17}{17}(-2,2,3) = n_3$$

$\therefore$ Minsta avst. $= |\overrightarrow{P_0P}_{n_3}| = |n_3| = \underline{\sqrt{17}\ \text{l.e.}}$

$$P' = P - \overrightarrow{PP_0}_{n_3} = P - n_3 = (2,4,4)-(-2,2,3) =$$

$$= \underline{\underline{(4,2,1)}}$$

2. $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 4 & 4 & 5 & 5 \\ 1 & 4 & 4 & 3 & 4 \end{pmatrix} \begin{matrix} -2 & -1 \end{matrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 1 & -1 \\ 0 & 3 & 3 & 1 & 1 \end{pmatrix} \begin{matrix} \\ 3 \\ 2 \end{matrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 6 & 6 & 3 & -3 \\ 0 & 6 & 6 & 2 & 2 \end{pmatrix} \begin{matrix} \\ -1 \\ \end{matrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 6 & 6 & 3 & -3 \\ 0 & 0 & 0 & -1 & 5 \end{pmatrix} \begin{matrix} \\ \frac{1}{3} \\ -1 \end{matrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix} \begin{matrix} -1 & -2 \end{matrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 13 \\ 0 & 2 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix} \begin{matrix} \\ \frac{1}{2} \\ \end{matrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 13 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix} \begin{matrix} \\ -1 \\ \end{matrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 11 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix} = T$

(a) $A \Leftrightarrow T$ där kolumn 1, 2 & 4 pivotkolumner

$\Rightarrow \text{Kolonn}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \right\} = \mathbb{R}^3$

$\text{rang}(A) = 3$

(b) $A\mathbf{x} = \mathbf{0} \Rightarrow (A \mid \mathbf{0}) \Leftrightarrow \ldots \Leftrightarrow (T \mid \mathbf{0}) =$

$= \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 11 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1 + 11x_5 = 0 \\ x_2 + x_3 + 2x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases}$

$\begin{matrix} x_3 = s \\ x_5 = t \\ \Rightarrow \end{matrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -11t \\ -s-2t \\ s \\ 5t \\ t \end{pmatrix} = s \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -11 \\ -2 \\ 0 \\ 5 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$

$\therefore \text{Noll}(A) = \text{Span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -11 \\ -2 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}$

$\text{nolldim}(A) = 2$

(c) Vet att: A har vänsterinvers $\Leftrightarrow$

$\Leftrightarrow$ A:s kolumner linjärt oberoende ($\Leftrightarrow$ massa andra påst.)

$\therefore$ A har ej vänsterinvers

(d) Vet att: A har högerinvers $\Leftrightarrow$

$\Leftrightarrow$ $A\mathbf{x} = \mathbf{b}$ lösbar $\forall \mathbf{b} \in \mathbb{R}^3$ ($\Leftrightarrow$ massa andra påståenden)

Detta stämmer då vi har ett pivotelement i varje rad.

$\therefore$ A har högerinvers

3. (a) <u>Sant</u>, $\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}$ dvs $\theta = \frac{\pi}{2}$

Vet $\mathbf{v} \perp \mathbf{u} \times \mathbf{v} \Rightarrow |\mathbf{v} \times (\mathbf{u} \times \mathbf{v})| = |\mathbf{v}||\mathbf{u} \times \mathbf{v}| =$

$= |\mathbf{v}||\mathbf{u}||\mathbf{v}| \sin \frac{\pi}{2} = |\mathbf{v}|^2 |\mathbf{u}|$

(b) <u>Falskt</u>, Kan finnas ∞ många lösn.

t.ex. $\begin{cases} x + y = 0 \\ 2x + 2y = 0 \\ 3x + 3y = 0 \end{cases}$

(c) <u>Falskt</u>, $\mathbb{O}$ ej ortogonal matris

(d) <u>Falskt</u>, $\lambda = 1$ kan också vara egenvärde till både A och A^2.

(Om $A = I$, så $I^2 = I$ men I har endast egenvädet $\lambda = 1$.)

(e) <u>Sant</u>, Om $z_0^n = 1$ så $|z_0|^n = 1 \Rightarrow |z_0| = 1$

$\Rightarrow z_0 \bar{z}_0 = |z_0|^2 = 1$. Om vi mult. båda sidor i $z_0 z_0^{n-1} = 1$ med $\bar{z}_0$ får vi:

$$\bar{z}_0 z_0 z_0^{n-1} = \bar{z}_0 \Leftrightarrow \bar{z}_0 = z_0^{n-1}$$

(f) <u>Falskt</u>, T.ex. byta plats på två rader är en elementär radoperation men determ. är alternerande.

4. $$\frac{4+8i}{1+i} = \frac{(1-i)(4+8i)}{(1-i)(1+i)} = \frac{4+8+i(8-4)}{2} = 6+2i$$

$$\frac{1+21i}{1+i} = \frac{(1-i)(1+21i)}{(1-i)(1+i)} = \frac{1+21+i(21-1)}{2} = 11+10i$$

$$p(z)=0 \iff z^2-(6+2i)z+11+10i=0 \iff$$

$$\iff z^2-2(3+i)z+(3+i)^2-(3+i)^2+11-10i=0 \iff$$

$$\iff (z-(3+i))^2 = 9+6i-1-11-10i \iff$$

$$\iff (z-(3+i))^2 = -3-4i \quad (*)$$

Låt $w = z-3-i$. Då $(*) \iff w^2 = -3-4i$

Antag $w = u+iv$, där $u, v \in \mathbb{R}$. Då

$$w^2 = u^2 - v^2 + i\cdot 2uv \overset{\text{vill}}{=} -3-4i$$

Därtill: $|w^2| = |-3-4i| \iff u^2+v^2 = \sqrt{9+16} = 5$

Sammantaget:

$$\begin{cases} u^2 - v^2 = -3 \\ u^2+v^2 = 5 \\ 2uv = -4 \end{cases} \Rightarrow 2u^2 = 2 \iff u = \pm 1$$

$$u_1 = 1 \Rightarrow v_1 = -\frac{2}{u_1} = -2 \Rightarrow w_1 = u_1 + iv_1 = 1-2i$$

$$u_2 = -1 \Rightarrow v_2 = -\frac{2}{u_2} = 2 \Rightarrow w_2 = u_2 + iv_2 = -1+2i$$

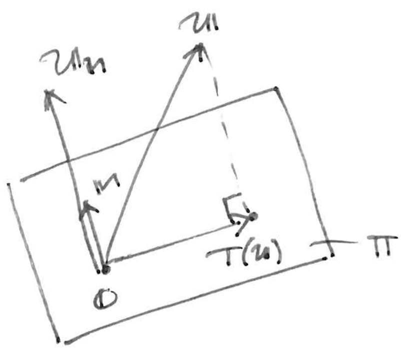
$$\therefore\ z_1 = w_1 + 3 + i = \underline{\underline{4-i}}$$

$$z_2 = w_2 + 3 + i = \underline{\underline{2+3i}}$$

$$p(z) = (1+i)\underline{\underline{(z-(4-i))(z-(2+3i))}}$$

5. $\pi : 3x - y + z = 0 \Rightarrow$

$\Rightarrow \mathbf{n} = (3, -1, 1)$



Ser att:

$$T(\mathbf{u}) = \mathbf{u} - \mathbf{u}_{\mathbf{n}} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}$$

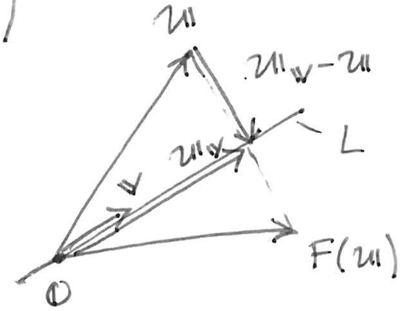
Detta ger:

$$T(\mathbf{e}_1) = \mathbf{e}_1 - \frac{\mathbf{e}_1\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{3}{11}\begin{pmatrix}3\\-1\\1\end{pmatrix} = \frac{1}{11}\begin{pmatrix}2\\3\\-3\end{pmatrix}$$

$$T(\mathbf{e}_2) = \mathbf{e}_2 - \frac{\mathbf{e}_2\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{(-1)}{11}\begin{pmatrix}3\\-1\\1\end{pmatrix} = \frac{1}{11}\begin{pmatrix}3\\10\\1\end{pmatrix}$$

$$T(\mathbf{e}_3) = \mathbf{e}_3 - \frac{\mathbf{e}_3\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{1}{11}\begin{pmatrix}3\\-1\\1\end{pmatrix} = \frac{1}{11}\begin{pmatrix}-3\\1\\10\end{pmatrix}$$

$$\therefore\ T(\mathbf{x}) = A\mathbf{x} \quad \text{där} \quad A = \frac{1}{11}\begin{pmatrix}2 & 3 & -3\\3 & 10 & 1\\-3 & 1 & 10\end{pmatrix}$$



$L: \quad x = y = \frac{z}{-1} \Rightarrow \mathbf{v} = (1, 1, -1)$

Ser att:

$$F(\mathbf{u}) = \mathbf{u} - 2(\mathbf{u}_{\mathbf{v}} - \mathbf{u}) = 2\mathbf{u}_{\mathbf{v}} - \mathbf{u} =$$

$$= 2\cdot\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} - \mathbf{u}$$

Detta ger:

$$F(\mathbf{e}_1) = 2\cdot\frac{\mathbf{e}_1\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} - \mathbf{e}_1 = \frac{2}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1\\2\\-2\end{pmatrix}$$

$$F(\mathbf{e}_2) = 2\cdot\frac{\mathbf{e}_2\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} - \mathbf{e}_2 = \frac{2}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$

$$F(\mathbb{e}_3) = 2\cdot\frac{\mathbb{e}_3\cdot\mathbb{v}}{|\mathbb{v}|^2}\mathbb{v} - \mathbb{e}_3 = -\frac{2}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\-2\\-1\end{pmatrix}$$

$\therefore$ $F(\mathbb{x}) = B\mathbb{x}$ där $B = \frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\2 & -1 & -2\\-2 & -2 & -1\end{pmatrix}$

Sammantaget:

$F \circ T$ linjär $\iff$ $F\circ T(\mathbb{x}) = C\mathbb{x}$

Å andra sidan:

$$F\circ T(\mathbb{x}) = F(T(\mathbb{x})) = F(A\mathbb{x}) = B(A\mathbb{x}) = BA\mathbb{x}$$

$$\Rightarrow C = BA = \frac{1}{33}\begin{pmatrix}-1 & 2 & -2\\2 & -1 & -2\\-2 & -2 & -1\end{pmatrix}\begin{pmatrix}2 & 3 & -3\\3 & 10 & 1\\-3 & 1 & 10\end{pmatrix} =$$

$$= \underline{\underline{\frac{1}{33}\begin{pmatrix}10 & 15 & -15\\7 & -6 & -27\\-7 & -27 & -6\end{pmatrix}}}$$

6. $AX = BX \iff AX - BX = \mathbf{0} \iff (A-B)X = \mathbf{0}$

Om $X = (\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3)$ så $(A-B)X = \mathbf{0} \iff$

$\iff (A-B)(\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3) = (\mathbf{0} \ \mathbf{0} \ \mathbf{0}) \iff$

$\iff ((A-B)\mathbf{x}_1 \ (A-B)\mathbf{x}_2 \ (A-B)\mathbf{x}_3) = (\mathbf{0} \ \mathbf{0} \ \mathbf{0})$

$$(A-B)\mathbf{x}_i = \mathbf{0} \Rightarrow (A-B \,|\, \mathbf{0}) = \left(\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \begin{matrix} \\ \\ -2 \quad -1 \end{matrix} \iff$$

$$\iff \begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \begin{matrix} \frac{1}{2} \\ \\ \\ \end{matrix} \iff \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$\Rightarrow \mathbf{x}_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{pmatrix} = \{x_{3i} = t_i\} = \begin{pmatrix} -2t_i \\ -2t_i \\ t_i \end{pmatrix}, \quad t_i \in \mathbb{R}, \ i = 1,2,3$$

$$\Rightarrow X = \begin{pmatrix} -2t_1 & -2t_2 & -2t_3 \\ -2t_1 & -2t_2 & -2t_3 \\ t_1 & t_2 & t_3 \end{pmatrix} \overset{\text{vill}}{=} \begin{pmatrix} -2t_1 & -2t_1 & t_1 \\ -2t_2 & -2t_2 & t_2 \\ -2t_3 & -2t_3 & t_3 \end{pmatrix} = X^T$$

$$\Rightarrow \begin{cases} -2t_1 = -2t_2 \\ t_1 = -2t_3 \\ t_2 = -2t_3 \end{cases} \overset{t_3 = s}{\Longrightarrow} \begin{cases} t_1 = -2s \\ t_2 = -2s \\ t_3 = s \end{cases}$$

$$\therefore X = \begin{pmatrix} 4s & 4s & -2s \\ 4s & 4s & -2s \\ -2s & -2s & s \end{pmatrix} = \underline{\underline{s \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}}}, \quad s \in \mathbb{R}$$

7. (a) <u>Definition</u>. Låt A typ $n \times n$. $\mathbf{x} \in \mathbb{R}^n$ egenvektor till A om $\mathbf{x} \neq \mathbf{0}$ och det finns något tal $\lambda \in \mathbb{R}$ så att $A\mathbf{x} = \lambda \mathbf{x}$.

(b) <u>Definition</u>: $p_A(\lambda) := \det(\lambda I - A)$ kallas för det <u>karakteristiska polynomet</u> för A.

($p_A(\lambda) := \det(A - \lambda I)$ går också bra)

(c) C har ej egenvärdet $0 \Leftrightarrow C$ inverterbar

Detta ger: $AC = CB \Leftrightarrow B = C^{-1}AC$

Vill visa att $B = C^{-1}AC$ & A har samma egenvärden.

Detta är Sats 6 (ii), s. 259 i Sparr (föreläsn. 24) och gäller då:

$$p_B(\lambda) = \det(\lambda I - B) = \det(\lambda C^{-1}C - C^{-1}AC) =$$

$$= \det\left(C^{-1}(\lambda I - A)C\right) = \det(C^{-1})\det(\lambda I - A)\det(C) =$$

$$= \det(C^{-1})\det(C)\, p_A(\lambda) = \det(C^{-1}C)\, p_A(\lambda) =$$

$$= \det(I)\, p_A(\lambda) = p_A(\lambda)$$

8. Se anteckningar på kurshemsidan