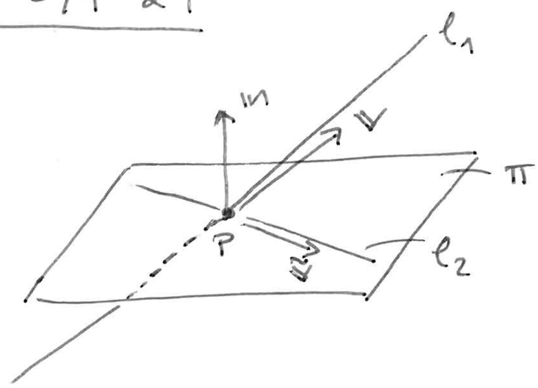
MVE670, Linjär algebra F/TM

Lösningar omtenta 1, 3/1-24



1. Låt $\pi : 3x - y + 2z = -8$

$$\ell_1 : \begin{cases} x = -1 + t \\ y = 3 - t \\ z = 4 + 3t \end{cases}, \quad t \in \mathbb{R}$$

då $\mathbf{v} = (1, -1, 3)$

och $\mathbf{n} = (3, -1, 2)$

Sökt: $\ell_2 : P + t\tilde{\mathbf{v}}$. Då P skär.pkt. mellan ℓ_1 och π.

$$\Rightarrow 3(-1+t) - (3-t) + 2(4+3t) = -8 \Leftrightarrow$$

$$\Leftrightarrow -3 + 3t - 3 + t + 8 + 6t = -8 \Leftrightarrow 2 + 10t = -8$$

$$\Leftrightarrow 10t = -10 \Leftrightarrow t = -1$$

$$\therefore P = (-1-1, 3+1, 4-3) = (-2, 4, 1)$$

$\tilde{\mathbf{v}}$ är parallell med $\mathbf{v}$:s ort. proj. på π =

$$= \mathbf{v} - \mathbf{v}_{\mathbf{n}} = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \frac{(1,-1,3)\cdot(3,-1,2)}{3^2 + (-1)^2 + 2^2}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \frac{10}{14}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 \\ -7 \\ 21 \end{pmatrix} - \frac{1}{7}\begin{pmatrix} 15 \\ -5 \\ 10 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7-15 \\ -7+5 \\ 21-10 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -8 \\ -2 \\ 11 \end{pmatrix}$$

$$\Rightarrow \tilde{\mathbf{v}} = (8, 2, -11)$$

$$\therefore \ell_2 : (-2, 4, 1) + t(8, 2, -11), \quad t \in \mathbb{R}$$

2. (a) $\mathbb{b} \notin \text{Kolonn}(A)$ om ekv. syst. $A\mathbb{x} = \mathbb{b}$ saknar lösn.

$$(A \,\vdots\, \mathbb{b}) = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 2 & -1 & -4 \\ 0 & -2 & 3 & 6 \\ 2 & 3 & -3 & -1 \end{array}\right) \begin{array}{c} -2 \\ 1 \end{array} \Leftrightarrow \begin{pmatrix} 2 & -1 & 1 & 5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 2 & 2 \\ 0 & 5 & -5 & -11 \end{pmatrix} \begin{array}{c} 5 \\ 1/2 \\ 2 \end{array} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} 2 & -1 & 1 & 5 \\ 0 & 10 & -5 & -20 \\ 0 & 0 & 1 & 1 \\ 0 & 10 & -10 & -22 \end{pmatrix} \begin{array}{c} -1 \quad 1/5 \end{array} \Leftrightarrow \begin{pmatrix} 2 & -1 & 1 & 5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -5 & -2 \end{pmatrix} \begin{array}{c} 5 \end{array} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} 2 & -1 & 1 & 5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \leftarrow 0 = 3 \; \lightning \quad \text{Går ej!}$$

$\therefore$ $A\mathbb{x} = \mathbb{b}$ saknar lösn. så $\mathbb{b} \notin \text{Kolonn}(A)$

(b) Vill lösa $A^T A \mathbb{x} = A^T \mathbb{b}$.

$$A^T A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 2 & -2 & 3 \\ 1 & -1 & 3 & -3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & 3 \\ 2 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 4 & -4 \\ 4 & 18 & -18 \\ -4 & -18 & 20 \end{pmatrix}$$

$$A^T \mathbb{b} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 2 & -2 & 3 \\ 1 & -1 & 3 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -28 \\ 30 \end{pmatrix}$$

$$(A^T A \,\vdots\, A^T \mathbb{b}) = \left(\begin{array}{ccc|c} 8 & 4 & -4 & 8 \\ 4 & 18 & -18 & -28 \\ -4 & -18 & 20 & 30 \end{array}\right) \begin{array}{c} 1/4 \\ 1/2 \\ 1/2 \end{array} \Leftrightarrow \begin{pmatrix} 2 & 1 & -1 & 2 \\ 2 & 9 & -9 & -14 \\ -2 & -9 & 10 & 15 \end{pmatrix} \begin{array}{c} -1 \quad 1 \end{array}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & 8 & -8 & -16 \\ 0 & -8 & 9 & 17 \end{pmatrix} \begin{array}{c} 1 \quad 1/8 \end{array} \Leftrightarrow \begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{c} 1 \end{array} \Leftrightarrow \begin{pmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{c} -1 \end{array}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{c} 1/2 \end{array} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\therefore$ $\mathbb{x} = (2, -1, 1)$

3. (a) <u>Sant</u>, $|\mathbf{u}\cdot\mathbf{v}|^2 + |\mathbf{u}\times\mathbf{v}|^2 = \big||\mathbf{u}||\mathbf{v}|\cos\theta\big|^2 + \big||\mathbf{u}||\mathbf{v}|\sin\theta\big|^2 =$

$= |\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta + |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta = |\mathbf{u}|^2|\mathbf{v}|^2(\sin^2\theta + \cos^2\theta) =$

$= |\mathbf{u}|^2|\mathbf{v}|^2$

(b) <u>Sant</u>, Om $A^T = A^{-1}$, $B^T = B^{-1}$ så

$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$

(c) <u>Falskt</u>, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ inv.bar då $\det(A) = 1 \neq 0$

men har endast egenvektorn $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) <u>Sant</u>, Om $f, g : \mathbb{R} \to \mathbb{R}$ kont. och udda

(dvs $f(-x) = -f(x)$, $g(-x) = -g(x)$) och $a, b \in \mathbb{R}$

så $(af + bg)(-x) = af(-x) + bg(-x) =$

$= -af(x) - bg(x) = -(af(x) + bg(x))$

(e) <u>Falskt</u>, T.ex. $f(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\mathbf{x}$ så

$f(1,0,0) = f(2,0,-1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ så f ej injektiv

(f) <u>Sant</u>, $\bar{z} = \dfrac{3}{z} \Leftrightarrow z\bar{z} = 3 \Leftrightarrow |z|^2 = 3 \Leftrightarrow$

$\Leftrightarrow |z| = \sqrt{3} < 2$

4. (a) $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & a \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 2-2a \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ \frac{2-2a}{3} \end{pmatrix} \overset{\text{vill}}{=} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$\Rightarrow \frac{2-2a}{3} = -2 \Leftrightarrow 2-2a = -6 \Leftrightarrow 2a = 8 \Leftrightarrow \underline{\underline{a=4}}$

(b) $p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} {}^{+}\lambda-3 & {}^{-}-2 & {}^{+}-2 \\ -2 & \lambda-2 & {}^{-}0 \\ -2 & 0 & {}^{+}\lambda-4 \end{vmatrix} =$

$= (-2)\begin{vmatrix} -2 & \lambda-2 \\ -2 & 0 \end{vmatrix} + (\lambda-4)\begin{vmatrix} \lambda-3 & -2 \\ -2 & \lambda-2 \end{vmatrix} =$

$= (\lambda-4)\big((\lambda-3)(\lambda-2)-4\big) + (-2)\cdot 2(\lambda-2) =$

$= (\lambda-4)(\lambda-3)(\lambda-2) - 4(\lambda-4) - 4(\lambda-2) =$

$= (\lambda-4)(\lambda-3)(\lambda-2) - 4(\lambda-4+\lambda-2) =$

$= (\lambda-4)(\lambda-3)(\lambda-2) - 8(\lambda-3) =$

$= (\lambda-3)\big((\lambda-4)(\lambda-2)-8\big) = (\lambda-3)(\lambda^2-6\lambda) =$

$= \lambda(\lambda-3)(\lambda-6) \Rightarrow \lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 6$

(Alt. lösn.: Vet från (a) att $\lambda = 3$ rot, så beräkna $p_A(\lambda)$ som 3:e gradspolynom och polynomdividera med $\lambda - 3$.)

$\underline{\lambda_1 = 0}$: $(\lambda I - A \,|\, 0) = \left(\begin{array}{ccc|c} -3 & -2 & -2 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -4 & 0 \end{array}\right) \Leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} \begin{matrix} -3 & -1 \end{matrix}$

$\Leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \begin{matrix} 1 & -1 \end{matrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix} -2t \\ 2t \\ t \end{pmatrix},\ t \neq 0$

$\underline{\lambda_2 = 3}$: Vet från (a) att $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} t \; , \; t \neq 0$

$\underline{\lambda_3 = 6}$: $(\lambda I - A \,|\, 0) = \left(\begin{array}{ccc|c} 3 & -2 & -2 & 0 \\ -2 & 4 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right) \Leftrightarrow$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & -2 & -2 & 0 \end{pmatrix} \; -1 \;\; -3 \; \Leftrightarrow \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 4 & -2 & 0 \end{pmatrix} \; 1 \;\; -2 \; \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix} t \\ t/2 \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tilde{t} \; , \; \tilde{t} \neq 0$$

$\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ ortogonala $\Longrightarrow$

$\Longrightarrow \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ ON-bas

$$5.\quad |z-2i| = |1+2iz| \overset{\geq 0}{\iff} |z-2i|^2 = |1+2iz|^2$$

$$VL = |z-2i|^2 = (z-2i)\overline{(z-2i)} = (z-2i)(\bar{z}+2i) =$$

$$= z\bar{z} + 2iz - 2i\bar{z} - (2i)^2 = \underbrace{|z|^2}_{=1} + 2iz - 2i\bar{z} + 4 =$$

$$= 5 + 2iz - 2i\bar{z}$$

$$HL = |1+2iz|^2 = (1+2iz)\overline{(1+2iz)} =$$

$$= (1+2iz)(1-2i\bar{z}) = 1 - 2i\bar{z} + 2iz - (2i)^2 \underbrace{|z|^2}_{=1} =$$

$$= 1 + 2iz - 2i\bar{z} + 4 = 5 + 2iz - 2i\bar{z}$$

$$\therefore\quad |z-2i| = |1+2iz|$$

6. $F\begin{pmatrix}1\\1\\-1\end{pmatrix}+F\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}-1\\-2\\1\end{pmatrix}+\begin{pmatrix}3\\6\\-3\end{pmatrix}\overset{\text{linjär}}{\Longleftrightarrow} F\left(\begin{pmatrix}1\\1\\-1\end{pmatrix}+\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)=\begin{pmatrix}2\\4\\-2\end{pmatrix}$

$\Longleftrightarrow F\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}2\\4\\-2\end{pmatrix}\overset{\text{linjär}}{\Longleftrightarrow} 2F\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\4\\-2\end{pmatrix}\Longleftrightarrow F(\mathbf{e}_1)=\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$F(\mathbf{e}_2)=F\left(\begin{pmatrix}-1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\overset{\text{linjär}}{=}F\begin{pmatrix}-1\\1\\0\end{pmatrix}+F\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}7\\0\\3\end{pmatrix}+\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}8\\2\\2\end{pmatrix}$

$F(\mathbf{e}_3)=F\left(\begin{pmatrix}1\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\overset{\text{linjär}}{=}F\begin{pmatrix}1\\-1\\1\end{pmatrix}-F(\mathbf{e}_1)+F(\mathbf{e}_2)=$

$=\begin{pmatrix}3\\6\\-3\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}+\begin{pmatrix}8\\2\\2\end{pmatrix}=\begin{pmatrix}10\\6\\0\end{pmatrix}$

$\therefore\ F(\mathbf{x})=A\mathbf{x}$ då $A=\begin{pmatrix}F(\mathbf{e}_1) & F(\mathbf{e}_2) & F(\mathbf{e}_3)\end{pmatrix}=\begin{pmatrix}1&8&10\\2&2&6\\-1&2&0\end{pmatrix}$

V_F = värdemängd F = Kolonn(A)

$\begin{pmatrix}1&8&10\\2&2&6\\-1&2&0\end{pmatrix}\begin{matrix}-2 & 1\end{matrix}\Longleftrightarrow\begin{pmatrix}1&8&10\\0&-14&-14\\0&10&10\end{pmatrix}\Longleftrightarrow\begin{pmatrix}1&8&10\\0&1&1\\0&0&0\end{pmatrix}$

$\Rightarrow V_F=$ Kolonn$(A)=$ Span$\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix},\begin{pmatrix}8\\2\\2\end{pmatrix}\right\}=$ Span$\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix},\begin{pmatrix}4\\1\\1\end{pmatrix}\right\}=$

$=$ planet π: $s\begin{pmatrix}1\\2\\-1\end{pmatrix}+t\begin{pmatrix}4\\1\\1\end{pmatrix}$, $s,t\in\mathbb{R}$ på parameterform

$\mathbf{n}=\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}4\\1\\1\end{pmatrix}=\begin{pmatrix}3\\-5\\-7\end{pmatrix}$ så planets ekv. på normalform

blir π: $3x-5y-7z=0$

$\exists\,\mathbf{x}\in\mathbb{R}^3$ s.a. $F(\mathbf{x})=(1,1,1)\Longleftrightarrow(1,1,1)\overset{?}{\in}V_F\Longleftrightarrow$

$\Longleftrightarrow(1,1,1)\overset{?}{\in}\pi$. Vi kollar: $3-5-7\neq0$

$\therefore\ (1,1,1)\notin V_F$

7. (a) Se Holmåker, Definition 2.1

(b) <u>Bevis</u>: $\|u+v\|^2 + \|u-v\|^2 =$

$$= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle =$$

$$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle +$$

$$+ \langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle =$$

$$= 2\langle u,u \rangle + 2\langle v,v \rangle = 2\|u\|^2 + 2\|v\|^2$$