Solutions to some recommended exercises where full solutions are *not* available in Dobrow's appendix

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1 Dobrow chapter 1

17. Using that the expectation of the Poisson distribution with parameter λ is λ we may compute for example

$$
\begin{aligned}
\mathrm{E}(X \mid X > 2) &= \frac{\sum_{k=3}^{\infty} k \,\mathrm{Poisson}(k;3)}{P(X>2)} \\
&= \frac{\sum_{k=0}^{\infty} k \,\mathrm{Poisson}(k;3) - \mathrm{Poisson}(1;3) - 2\,\mathrm{Poisson}(2,3)}{1 - P(X=0) - P(X=1) - P(X=2)} \\
&= \frac{3 - e^{-3}(3 + 2\frac{9}{2})}{1 - e^{-3}(1 + 3 + \frac{9}{2})} = 4.165246
\end{aligned}
$$

26.

$$
\begin{aligned}
P(Y < 2) &= \int_0^{\infty} P(Y < 2 \mid X = x) x e^{-x}\, dx \\
&= \int_0^2 x e^{-x}\, dx + \int_2^{\infty} \frac{2}{x} x e^{-x}\, dx \\
&= 1 - 3e^{-2} + 2e^{-2} = 1 - e^{-2}
\end{aligned}
$$

33. A possible code is

```
nSims <- 10000
simlist <- rep(0, nSims)
for (i in 1:nSims) {
  count <- 0
  while (TRUE) {
    card <- sample(1:52, 1)
    count <- count + 1
    if (card<=4) break;
```

```
  }
  simlist[i] <- count
}
print(mean(simlist))
print(var(simlist))
```

2 Dobrow Chapter 2

11. (a) We get for the elements of the transition matrix that, for $0 \leq i, j \leq 5$,

$$P_{ij} = \binom{5-i}{j-i} \left(\frac{1}{6}\right)^{j-i} \left(\frac{5}{6}\right)^{5-j}.$$

(b) Using, e.g., R, we get $P^3_{0,5} = 0.01327$.

(c) After 100 throws, we expect that we will have been able to obtain 5 sixes, no matter how many sixes we start with. Thus P^{100} should consist of zeroes, except for the last column which should consist of 1's.

3 Dobrow Chapter 8

5. We provide answers first using theory for multivariate normal distributions, and then more direct computation.

First, we have seen that (B_s, B_t) has a bivariate normal density. We have $\mathrm{E}(B_s) = \mathrm{E}(B_t) = 0$, $\mathrm{Var}(B_s) = \mathrm{Cov}(B_s, B_t) = s$, and $\mathrm{Var}(B_t) = t$, so

$$(B_s, B_t) \sim \text{Normal}\left((0,0), \begin{bmatrix} s & s \\ s & t \end{bmatrix}\right).$$

In general, for vectors u and v with

$$(u, v) \sim \text{Normal}\left((\mu_u, \mu_v), \begin{bmatrix} \Sigma_{uu} & \Sigma_{uv} \\ \Sigma_{vu} & \Sigma_{vv} \end{bmatrix}\right)$$

we have

$$u \mid v \sim \text{Normal}\left(\mu_u + \Sigma_{uu}\Sigma_{uv}^{-1}(v - \mu_v), \Sigma_{uu} - \Sigma_{uv}\Sigma_{vv}^{-1}\Sigma_{vu}\right).$$

In our case $\mu_u = \mu_v = 0$, $\Sigma_{uu} = \Sigma_{uv} = \Sigma_{vu} = s$, and $\Sigma_{vv} = t$ so

$$B_s \mid B_t = y \sim \text{Normal}\left(\frac{s}{t}y, s - \frac{s^2}{t}\right).$$

Not using this theory, we note that

$$\mathrm{P}(B_s = x, B_t = y) = \mathrm{P}(B_s = x, B_t - B_s = y - x)$$

where

$$\begin{aligned} B_s &\sim \mathrm{Normal}(0, s) \\ B_{t-s} &\sim \mathrm{Normal}(0, t-s) \end{aligned}$$

are independent. Thus the joint density is

$$\begin{aligned} \pi(x, y) &= \mathrm{Normal}(x; 0, s)\,\mathrm{Normal}(y - x; 0, t - s) \\ &= \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{1}{2s}x^2\right) \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{1}{2(t-s)}(y-x)^2\right) \\ &= \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2s}x^2 - \frac{1}{2(t-s)}(y-x)^2\right) \end{aligned}$$

For the conditional density we may compute for example

$$\begin{aligned} \mathrm{P}(B_s = x \mid B_t = y) &= \frac{\mathrm{P}(B_s = x, B_t = y)}{\mathrm{P}(B_t = y)} \\ &= \frac{\frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2s}x^2 - \frac{1}{2(t-s)}(y-x)^2\right)}{\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}y^2\right)} \\ &= \frac{1}{\sqrt{2\pi s(t-s)/t}} \exp\left(-\frac{1}{2\frac{s(t-s)}{t}} \left(x - \frac{s}{t}y\right)^2\right) \end{aligned}$$

leading to the same answer as above.