MVE550 2023 Lecture 11
Poisson Processes

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Where are we?

- In the beginning of the course, we defined a stochastic process as a collection $\{X_t, t \in I\}$ of random variables with a common state space S.
- So far, the set I has been the non-negative integers. We now move on to processes where I is a non-countable set, for example all positive real numbers, or all subsets of $\mathbb{R}^2$.
- Chapters 6 and 7 of Dobrow concern such stochastic processes where the state space S is discrete.
- In Chapter 8 of Dobrow we look at the situation when the random variables X_t are continuous variables.

Poisson distributions and Poisson processes

- A random variable with values $0, 1, 2, \ldots$ with a *Poisson distribution* can be used to model the *count* of events happening independently, within some time interval.
- We have seen that if $X \sim \text{Poisson}(\lambda)$ then $\pi(x) = \frac{\lambda^x}{x!} e^{-\lambda}$ and $\text{E}(X) = \lambda$, $\text{Var}(X) = \lambda$.
- A *Poisson process* models not only the count for a specific time interval, but also the exact time of every event.

Counting processes

- A *counting process* $\{N_t, t \in I\}$ is a stochastic process where $I = \mathbb{R}_0^+$, where the state space is the non-negative integers, and where $0 \leq s \leq t$ implies $N_s \leq N_t$.
- Informally, when $s < t$, $N_t - N_s$ counts the number of "events" in $(s, t]$.
- A realization of N_t is a function of t that is a right-continuous step function.

Poisson process: Definition 1

- A Poisson process $\{N_t\}_{t\geq 0}$ with parameter $\lambda > 0$ is a counting process fulfilling
  - $N_0 = 0$.
  - $N_t \sim \text{Poisson}(\lambda t)$ for all $t > 0$.
  - *Stationary increments:* $N_{t+s} - N_s$ has the same distribution as N_t for all $s > 0$, $t > 0$.
  - *Independent increments:* $N_t - N_s$ and $N_r - N_q$ are independent, when $0 \leq q < r \leq s < t$.
- Note: Not obvious that such a process exists.
- Note: $\text{E}(N_t) = \lambda t$. Thus what one is counting occurs with a *rate* of λ items per time unit.

Review: The exponential distribution

A random variable X with non-negative values as possible values has an exponential distribution with parameter λ if the density is

$$\pi(x) = \lambda e^{-\lambda x}.$$

The cumulative probability distribution is

$$F(x) = 1 - e^{-\lambda x}.$$

The expectation is $\frac{1}{\lambda}$. The variance is $\frac{1}{\lambda^2}$.

Memorylessness of the exponential distribution

- A random variable X is called *memoryless* if

$$P(X > s + t \mid X > s) = P(X > t)$$

  for all $s > 0$, $t > 0$.
- The exponential distribution is memoryless, and is the only memoryless continuous random variable.
- Consider the consequences of this when using the exponential as a model!

Poisson process: Definition 2

- Definition 2: Let $X_1, X_2, \ldots,$ be a sequence of iid exponential random variables with parmeter λ. Define $N_0 = 0$ and, for $t > 0$,

$$N_t = \max\{n : X_1 + \cdots + X_n \leq t\}.$$

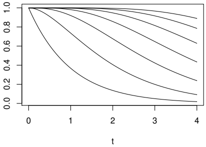
  Then $\{N_t\}_{t \geq 0}$ is a Poisson process with parameter λ.
- If we start with a Poisson process (def. 1) and let $X_1, X_2, \ldots$ be *inter-arrival times*, then they are independent exponentially distributed and N_t is given as above.
- Conversely, if we construct N_t as above, all properties of definition 1 are easily proved except that $N_t \sim \text{Poisson}(\lambda t)$: We discuss this below.
- The definition provides an easy way to simulate a Poisson process.
- We call $S_n = X_1 + \cdots + X_n$ the *arrival times* of the process.

Minimum and sum of independent exponentially distributed variables

- Define $M = \min(X_1, \ldots, X_n)$ where, independently for each i, $X_i \sim \text{Exponential}(\lambda_i)$. Then:
  - $M \sim \text{Exponential}(\lambda_1 + \cdots + \lambda_n)$.
  - $P(M = X_k) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}$.
- We will prove in an exercise: Let $S_n = X_1 + \cdots + X_n$ where, independently for each i, $X_i \sim \text{Exponential}(\lambda)$. Then $S_n \sim \text{Gamma}(n, \lambda)$.
- Using the distribution of S_n, one can complete the proof that a process defined with "Definition 2" is a Poisson process:

$$\Pr(N_t = k) = \Pr(S_k \leq t, S_k + X_{k+1} > t) = \cdots = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

Poisson process: Definition 3



- Plot shows, for each t, the probabilities of observing 0, 1 ... events: Derivatives of all curves at 0 are 0 except for the first curve.
- Third definition: A Poisson process $\{N_t\}_{t \geq 0}$ with parameter λ is a couting process fulfilling
  - $N_0 = 0$.
  - The process has stationary and independent increments.
  - $$\begin{aligned} P(N_h = 0) &= 1 - \lambda h + o(h) \\ P(N_h = 1) &= \lambda h + o(h) \\ P(N_h > 1) &= o(h) \end{aligned}$$
- All the three definitions of a Poission process are equivalent.

Example

At a hospital, births occur at a rate λ. For each birth there is a probability $p = 0.52$ that the child is a boy. The situation can be modelled in two ways:

- The counts c_1 of boys and c_2 of girls are modelled with two independent Poisson processes, $\left(N_t^{(1)}\right)_{t \geq 0}$ and $\left(N_t^{(2)}\right)_{t \geq 0}$, with parameters λp and $\lambda(1-p)$, respectively.
- The total number of births N is modelled with one Poisson process $(N_t)_{t \geq 0}$ and counts are then Binomially distributed given N:

$$c_1 \sim \text{Binomial}(N; p) \qquad c_2 = N - c_1$$

- Luckily, we can prove that these ways of modelling are equivalent.

Superposition and thinning

- LEMMA[1]: Let $\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$ be independent Poisson processes with parameters $\lambda p_1, \ldots, \lambda p_n$, respectively, where $p = (p_1, \ldots, p_n)$ is a probability vector. If $c = (c_1, \ldots, c_n)$ are the counts after time t (so that $c_i = N_t^{(i)}$), an equivalent model is

$$c \sim \text{Multinomial}(N, p)$$

  where $(N_t)_{t\geq 0}$ is a Poisson process with parameter λ.
- Proof on next page.
- Starting with one Poisson process and creating another by independently selecting arrivals with probability p and considering only those is called *thinning*.
- Starting with several independent Poisson processes and considering their joint counts is called *superposition*.

[1] A somewhat different treatment compared to Dobrow

Proof

- Using the model with independent Poisson processes, the probability of observing the count vector c after time t is (writing $N = c_1 + \cdots + c_n$)

$$\prod_{i=1}^{n} \mathrm{Poisson}(c_i; \lambda p_i t) = \prod_{i=1}^{n} e^{-\lambda p_i t} \frac{(\lambda p_i t)^{c_i}}{c_i!}$$

$$= \quad e^{-\lambda t} (\lambda t)^N \prod_{i=1}^{n} \frac{p_i^{c_i}}{c_i!} = e^{-\lambda t} \frac{(\lambda t)^N}{N!} \cdot \frac{N!}{c_1! \cdots c_n!} p_1^{c_1} \cdots p_n^{c_n}$$

$$= \quad \mathrm{Poisson}(N; \lambda t) \cdot \mathrm{Multinomial}\,(c; N, p)$$

- The process for N inherits independent and stationary increments from the sub-processes, so it follows it is also a Poisson process.

Uniformly distributed arrivals

- LEMMA[2]: Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. If we fix that $N_t = k$ and we select uniformly randomly one of these k arrivals, then its arrival time is uniformly distributed on the interval $[0, t]$.
- Proof on next page.
- Consequence: We can simulate a Poisson process on $[0, t]$ by first simulating N_t, and then simulating the N_t arrival times as independently uniformly distributed on the interval $[0, t]$.
- Consequence: When $N_t = k$ is fixed, the n'th arrival time has the same distribution as the n'th value among k independent uniformly distributed variables on $[0, t]$.

[2] A somewhat different treatment compared to Dobrow

Proof

$$
\begin{aligned}
& \Pr\left(S_k \geq s \mid k \text{ uniformly random in } \{1, \ldots, n\}, N_t = n\right) \\
= \quad & \frac{1}{n} \sum_{k=1}^{n} \Pr\left(S_k \geq s \mid N_t = n\right) = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \Pr\left(N_s = j \mid N_t = n\right) \\
= \quad & \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \frac{\Pr\left(N_s = j\right) \Pr\left(N_{t-s} = n - j\right)}{\Pr\left(N_t = n\right)} \\
= \quad & \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} \frac{e^{-\lambda s}(\lambda s)^j / j! \cdot e^{-\lambda(t-s)}(\lambda(t-s))^{n-j}/(n-j)!}{e^{-\lambda t}(\lambda t)^n / n!} \\
= \quad & \frac{1}{n} \sum_{j=0}^{n-1} (n-j) \frac{n!}{j!(n-j)!} \left(\frac{s}{t}\right)^j \left(1 - \frac{s}{t}\right)^{n-j} \\
= \quad & \left[\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-j-1)!} \left(\frac{s}{t}\right)^j \left(1 - \frac{s}{t}\right)^{n-j-1}\right] \left(1 - \frac{s}{t}\right) \\
= \quad & 1 - \frac{s}{t}
\end{aligned}
$$