MVE550 2023 Lecture 12
Poisson processes, part 2
Dobrow chapter 6

Petter Mostad

Chalmers University

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Poisson processes: Short review from last lecture

- A Poisson process $\{N_t\}_{t \geq 0}$ is defined with the requirements
  - A counting process.
  - $N_0 = 0$
  - Stationary increments.
  - Independent increments.
  - Either stating that $X_t \sim \text{Poisson}(\lambda t)$, or stating that, for t small, there is maximum one event in $[0, t]$, and the probability of such an event is $t\lambda$.
- A Poisson process can also be defined by counting sums of inter-arrival times $X_i \sim \text{Exponential}(\lambda)$.
- The n'th arrival time S_n distributed as $\text{Gamma}(n, \lambda)$.
- Today:
  - More than one "type" of event.
  - More properties of event distributions.
  - Nonhomogeneous and spatial point processes.

Example

At a hospital, births occur at a rate λ. For each birth there is a probability $p = 0.52$ that the child is a boy. The situation can be modelled in two ways:

- The counts c_1 of boys and c_2 of girls are modelled with two independent Poisson processes, $\left\{N_t^{(1)}\right\}_{t\geq 0}$ and $\left\{N_t^{(2)}\right\}_{t\geq 0}$, with parameters λp and $\lambda(1-p)$, respectively.
- The total number of births N is modelled with one Poisson process $\{N_t\}_{t\geq 0}$ and counts are then Binomially distributed given N:

$$c_1 \sim \text{Binomial}(N; p) \qquad c_2 = N - c_1$$

- Luckily, we can prove that these ways of modelling are equivalent.

Superposition and thinning

- LEMMA[1]: Let $\left\{N_t^{(1)}\right\}_{t\geq 0}, \ldots, \left\{N_t^{(n)}\right\}_{t\geq 0}$ be independent Poisson processes with parameters $\lambda p_1, \ldots, \lambda p_n$, respectively, where $p = (p_1, \ldots, p_n)$ is a probability vector. If $c = (c_1, \ldots, c_n)$ are the counts after time t (so that $c_i = N_t^{(i)}$), an equivalent model is

$$c \sim \text{Multinomial}(N_t, p)$$

  where $\{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ.
- Proof on next page.
- Starting with one Poisson process and creating another by independently selecting arrivals with probability p and considering only those is called *thinning*.
- Starting with several independent Poisson processes and considering their joint counts is called *superposition*.

[1] A somewhat different treatment compared to Dobrow

Proof

- Using the model with independent Poisson processes, the probability of observing the count vector c after time t is (writing $N = c_1 + \cdots + c_n$)

$$\begin{aligned}\prod_{i=1}^{n} \mathrm{Poisson}(c_i; \lambda p_i t) &= \prod_{i=1}^{n} e^{-\lambda p_i t} \frac{(\lambda p_i t)^{c_i}}{c_i!} \\ &= e^{-\lambda t} (\lambda t)^N \prod_{i=1}^{n} \frac{p_i^{c_i}}{c_i!} = e^{-\lambda t} \frac{(\lambda t)^N}{N!} \cdot \frac{N!}{c_1! \cdots c_n!} p_1^{c_1} \cdots p_n^{c_n} \\ &= \mathrm{Poisson}(N; \lambda t) \cdot \mathrm{Multinomial}\,(c; N, p)\end{aligned}$$

- The process for N inherits independent and stationary increments from the sub-processes, so it follows it is also a Poisson process.

Uniformly distributed arrivals

- LEMMA[2]: Let $\{N_t\}_{t\geq 0}$ be a Poisson process with parameter λ. If we fix that $N_t = k$ and we select uniformly randomly one of the k arrivals, then its arrival time is uniformly distributed on the interval $[0, t]$.
- Proof on next page.
- Consequence: We can simulate a Poisson process on $[0, t]$ by first simulating N_t, and then simulating the N_t arrival times as independently uniformly distributed on the interval $[0, t]$.
- Consequence: When $N_t = k$ is fixed, the n'th arrival time for $n \leq k$ has the same distribution as the n'th value among k independent uniformly distributed variables on $[0, t]$.

[2] A somewhat different treatment compared to Dobrow

Proof

$$
\begin{aligned}
& \Pr\left(S_k \geq s \mid k \text{ uniformly random in } \{1, \ldots, n\}, N_t = n\right) \\
= \quad & \frac{1}{n} \sum_{k=1}^{n} \Pr\left(S_k \geq s \mid N_t = n\right) = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \Pr\left(N_s = j \mid N_t = n\right) \\
= \quad & \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \frac{\Pr\left(N_s = j\right) \Pr\left(N_{t-s} = n-j\right)}{\Pr\left(N_t = n\right)} \\
= \quad & \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} \frac{e^{-\lambda s} (\lambda s)^j / j! \cdot e^{-\lambda(t-s)} (\lambda(t-s))^{n-j} / (n-j)!}{e^{-\lambda t} (\lambda t)^n / n!} \\
= \quad & \frac{1}{n} \sum_{j=0}^{n-1} (n-j) \frac{n!}{j!(n-j)!} \left(\frac{s}{t}\right)^j \left(1 - \frac{s}{t}\right)^{n-j} \\
= \quad & \left[\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-j-1)!} \left(\frac{s}{t}\right)^j \left(1 - \frac{s}{t}\right)^{n-j-1}\right] \left(1 - \frac{s}{t}\right) \\
= \quad & 1 - \frac{s}{t}
\end{aligned}
$$

Spatial Poisson processes

- A collection of random variables $\{N_A\}_{A \subseteq \mathbb{R}^d}$ is a spatial Poisson process with parameter λ if
  - For each bounded set $A \subseteq \mathbb{R}^d$, N_A has a Poisson distribution with parameter $\lambda|A|$.
  - Whenever $A \subseteq B$, $N_A \leq N_B$ (i.e., spatial counting process).
  - Whenever A and B are disjoint sets, N_A and N_B are independent.
- Simulate by first simulating the total (Poisson distributed) and then place points independently uniformly within the area.
- A special case of a *point process*.
- One may use simulations to estimate properties such as the average distance to the nearest neighbour (or the third nearest neighbour or whatever).
- Quite useful models in practice.

Non-homogeneous Poisson processes

- A counting process $\{N_t\}_{t \geq 0}$ is a *non-homogeneous* Poisson process with intensity function $\lambda(t)$ if
  - $N_0 = 0$.
  - For $0 < s < t$,

$$N_t - N_s \sim \text{Poisson}\left(\int_s^t \lambda(x)\, dx\right)$$

  - It has independent increments.
- Again a very flexible and useful model in practice.
- One may have non-homogeneous spatial Poisson processes.