MVE550 2023 Lecture 14
Dobrow Chapter 7, part 2

Petter Mostad

Chalmers University

December 5, 2023

Review from last time:

- Continuous-time discrete state space Markov chains.
- The generator matrix Q, consisting of rates.
- Exponentially distributed holding times.
- Connection with $\tilde{P}$, the embedded discrete-time chain.
- The matrix transition function $P(t)$. $P'(t) = QP(t) = P(t)Q$.
- The exponential matrix e^A for a square matrix A, and its computation.
- $P(t) = e^{tQ}$.

Review: The matrix exponential

- For any square matrix A define the *matrix exponential* as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \frac{1}{24}A^4 + \dots$$

- The series converges for all square matrices A (we don't show this).
- Some important properties:
  - $e^0 = I$.
  - $e^A e^{-A} = I$.
  - $e^{(s+t)A} = e^{sA} e^{tA}$.
  - If $AB = BA$ then $e^{A+B} = e^A e^B = e^B e^A$.
  - $\frac{\partial}{\partial t} e^{tA} = A e^{tA} = e^{tA} A$.
- $P(t) = e^{tQ}$ is the unique solution to the differential equations $P'(t) = QP(t)$ for all $t \geq 0$ and $P(0) = I$.
- In `R` you may use `expm` from R package `expm` to compute exponential matrices.

Computing the matrix exponential

- Assume there exists an invertible matrix S and a matrix D such that $Q = SDS^{-1}$. Then (show!)
$$e^{tQ} = Se^{tD}S^{-1}$$
- If $D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_k \end{bmatrix}$ is a diagonal matrix, then (show!)
$$e^{tD} = \begin{bmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{t\lambda_k} \end{bmatrix}.$$
- Recall that if Q is *diagonalizable* it can be written as $Q = SDS^{-1}$ where D is diagonal with the eigenvalues along the diagonal, and S has the corresponding eigenvectors as columns.

Limiting and stationary distributions

- A probability vector v represents a *limiting distribution* if, for all states i and j,

$$\lim_{t \to \infty} P_{ij}(t) = v_j.$$

- A probability vector v represents a *stationary distribution*, if, for all $t \geq 0$,

$$v = vP(t)$$

- Note: v is a stationary distribution if and only if $0 = vQ$.
- A limiting distribution is a stationary distribution but a stationary distribution is not necessarily a limiting distribution.

Review of limiting result for discrete-time chains

- An ergodic Markov chain has a unique positive stationary distribution that is the limiting distribution.
- For discrete-time chains, v is stationary if $vP = v$ where P is the transition matrix.
- A discrete-time chain is ergodic if it is irreducible, aperiodic, and all states have finite expected return times.

Fundamental limit theorem for cont. time chains

- A continuous-time Markov chain is *irreducible* if for all i and j there exists a $t > 0$ such that $P_{ij}(t) > 0$.
- However, periodic continuous-time Markov chains do not exist: If $P_{ij}(t) > 0$ for some $t > 0$ then $P_{ij}(t) > 0$ for all $t > 0$.
- **Fundamental Limit Theorem**: Let $\{X_t\}_{t \geq 0}$ be a finite, irreducible, continuous-time Markov chain with transition function $P(t)$. Then there exists a unique positive stationary distribution vector v which is also the limiting distribution.
- The limiting distribution of such a chain can be found as the unique v satisfying $vQ = 0$.

Classification of finite-state continuous-time chains

- An absorbing communication class is one where there is zero probability (i.e., zero rate) of leaving it to other communication classes.
- For a finite-state continuous-time Markov chain (with finite holding time parameters) there are two possibilities:
  - The process is irreducible, and $P_{ij}(t) > 0$ for all $t > 0$ and all i, j.
  - The process contains one or more absorbing communication classes.

Absorbing states

- Assume $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain with k states. Assume the last state is absorbing and the rest are not. (They are then transient).
- We have that $q_k = 0$ and the entire last row must consist of zeros. We get
$$Q = \begin{bmatrix} V & * \\ \mathbf{0} & 0 \end{bmatrix}.$$
- Let F be the $(k-1)\times(k-1)$ matrix so that F_{ij} (with $i < k$, $j < k$) is the expected time spent in state j when the chain starts in i. We can shown that $F = -V^{-1}$ (see next page).
- F is called the *fundamental matrix*.
- Note that, if the chain starts in state i, the expected time until absorbtion is the sum of the i'th row of F. Thus the expected times until absorbtion are given by the matrix product $F1$ of F with a column of 1's.

Outline of proof (different from Dobrow's)

- Generally, define D as the matrix with $(1/q_1, \ldots, 1/q_k)$ along its diagonal, with all other entries zero. If there are no absorbing states

$$\tilde{P} = DQ + I$$

- Write A_- for a square matrix without its last row and column.
- If the last state is absorbing, so that $q_k = 0$, we get

$$\tilde{P}_- = D_- Q_- + I$$

- Let F' be the matrix where F'_{ij} is the expected *number of stays* in state j before absorbtion when starting in state i. As the lengths of stays and changes in states are independent, we get $F = F' D_-$.
- From the theory of Chapter 3, we have that $F' = (I - \tilde{P}_-)^{-1}$.
- We get

$$F = F' D_- = (I - \tilde{P}_-)^{-1} D_- = (-D_- Q_-)^{-1} D_- = (-Q_-)^{-1}.$$

Stationary distribution of the embedded chain

- The embedded chain of a continuous-time Markov chain: The discrete-time Markov chain where holding times are ignored.
- Stationary distributions for the embedded chain and for the continuous-time chain are generally not the same!
- However, there is a simple relationship: A probability vector π is a stationary distribution for a continuous-time Markov chain if and only if ψ is a stationary distribution for the embedded chain, where $\psi_j = C\pi_j q_j$ for a constant C making the entries of ψ add to 1.
- *Proof*: Using notation above, we have $\tilde{P} = DQ + I$. For any vector v we get $v\tilde{P} = vDQ + v$, so $v\tilde{P} = v$ if and only if $vDQ = 0$.