MVE550 2023 Lecture 15
Dobrow Chapter 7, part 3

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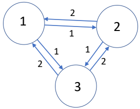
Review from last time:

- Continuous-time Markov chains, described by a generator matrix Q, consisting of rates.
- $P(t) = e^{tQ}$.
- Limiting and stationary distributions, the fundamental limit theorem.
- Solve $vQ = 0$ to find limiting distribution.
- Absorbing states, the fundamental matrix $F = -V^{-1}$.
- The stationary distribution of a continuous-time chain and its embedded chain are different, but related.

Global Balance

- 

Let $v = (v_1, v_2, v_3)$ be the stationary distribution. At v, the flow into a state must be equal to the flow out of that state.
- We get: $2v_1 = 2v_2 + 2v_3$, $3v_2 = 1v_1 + 2v_3$, and $4v_3 = 1v_1 + 1v_2$.
- Note that these are exactly the equations we get from $vQ = 0$:

$$(v_1, v_2, v_3)\begin{bmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{bmatrix} = 0$$

- This happens because $vQ = 0$ gives for each state j

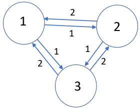
$$\sum_{i \neq j} v_i q_{ij} = v_j q_j$$

- These are called the global balance equations.
- Generalization (proof in Dobrow): If A is a set of states, then the long term rates of movement *into* and *out of* A are the same:

$$\sum_{i \notin A} \sum_{j \in A} v_i q_{ij} = \sum_{i \notin A} \sum_{j \in A} v_j q_{ji}$$

Local balance and time reversibility

- Local balance: A stronger condition: The flow between *every pair* of states is balanced. This is *not* true for all models!
- For the model above the local balance equations are



$$1v_1 = 2v_2,\ 1v_2 = 2v_3, \text{ and } 2v_3 = 1v_1.$$

- These are *not* satisfied by the stationary v! (Check it).
- An irreducible continuous-time Markov chain with stationary distribution v is said to be *time reversible* if for all i, j,

$$v_i q_{ij} = v_j q_{ji}$$

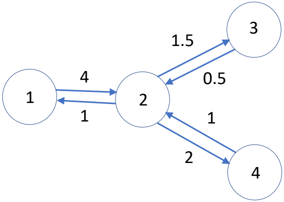
  which is in fact the *local balance* condition.
- Note: The rate of observed changes from i to j is the same as the rate of observed changes from j to i. Thus this is called time reversibility.
- Note that (similar to discrete chains): If a probability vector v satisfies local balance condition, then v is a stationary distribution. (Easy to show directly).

Markov processes with transition graphs that are trees

- **A tree is a graph that does not contain cycles**.
- Assume the transition graph of an irreducible continuous-time Markov chain is a tree.
- In a tree, any edge between two states divides all states into two groups (each on each side of the edge). Thus, the flow must be balanced across each edge.
- It follows that the Markov chain must satisfy the local balance condition, i.e., be time reversible, i.e., $v_i q_{ij} = v_j q_{ji}$ for all i and j.
- More formally, this can be proved using the generalized global balance property.
- Note that the process can be time reversible even if the transition graph is not a tree.

Example

- Consider the continuous-time Markov chain with transition graph



- As the transition graph is a tree, the chain is necessarily time reversible. We can find the stationary distribution by considering the local balance equations:

$$4v_1 = 1v_2, \qquad 1.5v_2 = 0.5v_3, \qquad 2v_2 = 1v_4$$

- Together with the equation $v_1 + v_2 + v_3 + v_4 = 1$ we easily get the limiting distribution

$$v = \left(\frac{1}{25}, \frac{4}{25}, \frac{12}{25}, \frac{8}{25} \right)$$

Birth-and-death processes

- A birth-and-death process is a continuous-time Markov chain where the state space is the set of nonnegative integers and transitions only occur to neighbouring integers.
- The process is necessarily time-reversible, as the transition graph is a tree (in fact, a line).
- We denote the rate of *births* from i to $i+1$ with λ_i, and the rate of *deaths* from i to $i-1$ with μ_i.
- The generator matrix is

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\mu_3+\lambda_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Provided $\sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} < \infty$, the unique stationary distribution is given by

$$v_k = v_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \text{for } k = 1, 2, \dots, \qquad v_0 = \left(1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}\right)^{-1}$$

Example

- The simplest example of a birth-and-death process is one where all birth rates λ_i and all death rates μ_i are the same values λ and μ, respectively.
- We get that

$$
\begin{aligned}
v_k &= v_0 \prod_{i=1}^{k} \frac{\lambda}{\mu} = v_0 \left(\frac{\lambda}{\mu}\right)^k \\
v_0 &= \left(\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k\right)^{-1} = \frac{1}{1 + \frac{\lambda}{\mu} + (\frac{\lambda}{\mu})^2 + \dots} = \frac{1}{1/(1 - \frac{\lambda}{\mu})} = 1 - \frac{\lambda}{\mu}
\end{aligned}
$$

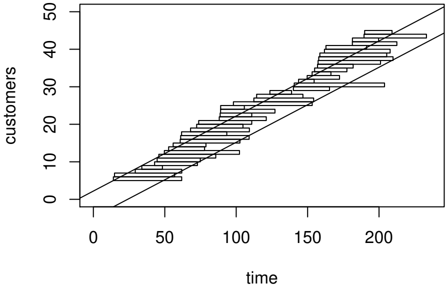
- We see that the limiting distribution is Geometric $\left(1 - \frac{\lambda}{\mu}\right)$.
- For example, the long-term average value of X_t will be

$$\frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{\lambda}{\mu - \lambda}$$

Queueing theory

- Birth-and-death processes are special cases of queues.
- In the more general theory of queues:
  - The arrival process ("births") need not be a Poisson process, with exponentially distributed inter-arrival times.
  - The service times in the system need not be exponentially distributed.
  - There can be many other generalizations, such as how many servers there are, how the servers work, how the lines work, etc.
- One can use notation $A/B/n$ where A denotes arrival process, B denotes service process, and n the number of servers.
- With this notation, our birth-and-death model above with constant birth and death rates is denoted $M/M/1$. (M means Markov).
- Generalize our results to the case $M/M/c$ by taking into account that some servers are idle when there are fewer than c customers.

Little's formula



Little's formula is valid when customers arrive at rate λ, and stay an expected time W. The left line represents the average arrival times of customers: It has slope λ. The right line represents the average departure time of customers. The horizontal distance between the lines is W. The vertical distance between the lines will be L, the average number of customers in the system. Thus

$$\lambda = \frac{L}{W}$$

Poisson subordination

- We may simulate from a continuous time finite state Markov chain by drawing the holding times from distributions Exponential(q_i), where q_i depends on the state i, and then use $\tilde{P}$.
- INSTEAD, always simulate holding times from Exponential(λ) where λ is large, and allow movement back to the same state.
- Matematical formulation: Given generator matrix Q. If $\lambda \geq \max(q_, \dots, q_k)$ then
  - $R = \frac{1}{\lambda}Q + I$ is a stochastic matrix.
  - We can write

$$P(t) = e^{tQ} = e^{-t\lambda I}e^{t\lambda R} = e^{-t\lambda}\sum_{k=0}^{\infty}\frac{(t\lambda R)^k}{k!} = \sum_{k=0}^{\infty} R^k \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

- Thus: To find the probability of going from i to j during time t:
  1. Condition on number of changes occurring $k \sim$ Poisson(λt).
  2. For each k, use k steps of discrete chain with transition matrix R.
- This provides a good way to compute e^{tQ}: Throw away terms where k is over some limit. Better accuracy than using definition of exponential matrix!
- Ths discrete chain has the same stationary distribution as the continuous chain.