MVE550 2023 Lecture 17
Dobrow Chapter 8, part 2

Petter Mostad

Chalmers University

December 12, 2023

Review: Brownian motion

- A process $\{B_t\}_{t \geq 0}$ where $B_t \sim \text{Normal}(0, t)$. No parameters.
- Independent normally distributed increments.
- Continuous paths that are nowhere differentiable.
- Connection to random walks: The Donsker principle.
- Gaussian processes.
- Restarting Brownian motions at stopping times.

The distribution of the first hitting time

- Given $a \neq 0$ what is the distribution of the first hitting time $T_a = \min\{t : B_t = a\}$?
- We prove below that

$$\frac{1}{T_a} \sim \text{Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$$

- Assuming that $a > 0$ and using that T_a is a stopping time we get for any $t > 0$ that $\Pr\left(B_{1/t} > a \mid T_a < 1/t\right) = \Pr\left(B_{1/t - T_a} > 0\right) = \frac{1}{2}$.
- We also have

$$\Pr\left(B_{1/t} > a \mid T_a < 1/t\right) = \frac{\Pr\left(B_{1/t} > a, T_a < 1/t\right)}{\Pr\left(T_a < 1/t\right)} = \frac{\Pr\left(B_{1/t} > a\right)}{\Pr\left(T_a < 1/t\right)}.$$

- It follows that $\Pr\left(T_a < 1/t\right) = 2\Pr\left(B_{1/t} > a\right)$ and so

$$\Pr\left(\frac{1}{T_a} < t\right) = 2\Pr\left(B_{1/t} < a\right) - 1 = 2\Pr\left(B_1 < a\sqrt{t}\right) - 1.$$

- Using $B_1 \sim \text{Normal}(0,1)$ and taking the derivative w.r.t. t we get the Gamma density above:

$$\pi_{1/T_a}(t) = 2\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(a\sqrt{t})^2\right) \frac{a}{2} t^{-1/2}.$$

Maximum of Brownian motion

- Define $M_t = \max_{0 \le s \le t} B_s$.
- We may compute for $a > 0$ (using result from previous page)

$$\Pr(M_t > a) = \Pr(T_a < t) = 2\Pr(B_t > a) = \Pr(|B_t| > a)$$

- Thus M_t has the same distribution as $|B_t|$, the absolute value of B_t.
- Example: What is the probability that $M_3 > 5$?
- Example: Find t such that $\Pr(M_t \le 4) = 0.9$.

Zeros of Brownian motion

- Let L be the *last zero* in $(0,1)$ of Brownian motion. (In other words, $L = \max\{t : 0 < t < 1, B_t = 0\}$. Then

$$L \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right).$$

- Outline of proof on next page.
- Consequence: Let L_t be the last zero in $(0, t)$. Then

$$L_t/t \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right).$$

- Note: The probability that Brownian motion has at least one zero in (r, t) for $0 \leq r < t$ is $1 - \Pr(L_t < r)$.
- Note: The cumulative distribution for the Beta density can be computed with the arcsin function:

$$\Pr(L_t < r) = \int_0^{r/t} \text{Beta}\left(s; \frac{1}{2}, \frac{1}{2}\right)\, ds = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{r}{t}}\right)$$

Outline of proof

$$
\begin{aligned}
\Pr(L > s) &= \int_{-\infty}^{\infty} \Pr(L > s \mid B_s = t)\,\mathrm{Normal}(t; 0, s)\,dt \\
&= 2\int_{-\infty}^{0} \Pr(L > s \mid B_s = t)\,\mathrm{Normal}(t; 0, s)\,dt \\
&= 2\int_{-\infty}^{0} \Pr(M_{1-s} > -t)\,\mathrm{Normal}(t; 0, s)\,dt \\
&= 2\int_{0}^{\infty} 2\Pr(B_{1-s} > t)\,\mathrm{Normal}(t; 0, s)\,dt \\
&= 4\int_{0}^{\infty}\int_{t}^{\infty} \mathrm{Normal}(r; 0, 1-s)\,\mathrm{Normal}(t; 0, s)\,dr\,dt \\
&= \ldots \\
&= \frac{1}{\pi}\int_{s}^{1} \frac{1}{\sqrt{x(1-x)}}\,dx \\
&= \int_{s}^{1} \mathrm{Beta}\left(x; \frac{1}{2}, \frac{1}{2}\right)\,dx
\end{aligned}
$$

Brownian bridge

- Define a Gaussian process X_t by conditioning Brownian motion B_t on $B_1 = 0$. Then X_t is a *Brownian bridge*.
- If $0 < s < t < 1$ then (B_s, B_t, B_1) is multivariate normal with

$$\mathrm{E}\left((B_s, B_t, B_1)\right) = (0, 0, 0), \quad \mathrm{Var}\left((B_s, B_t, B_1)\right) = \Sigma = \begin{bmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{bmatrix}.$$

  Conditioning on $B_1 = 0$ and using properties of the multivariate normal (or see Dobrow) we get $\mathrm{E}\left(X_t\right) = 0$ and

$$\mathrm{Cov}\left(X_s, X_t\right) = s - st.$$

- Define another Gaussian process with $Y_t = B_t - tB_1$. Then we see that $\mathrm{E}\left(Y_t\right) = 0$ and (when $0 < s < t < 1$)

$$\mathrm{Cov}\left(Y_s, Y_t\right) = s - st.$$

  It follows that this is identical to the Brownian bridge defined above.
- Example: Estimate by simulation: If a Brownian motion fulfills $B_1 = 0$, what is the probability that it has values below -1?

Brownian motion with a drift

- For real μ and $\sigma > 0$ define the Gaussian process X_t as

$$X_t = \mu t + \sigma B_t$$

  This is *Brownian motion with a drift*, and is often a more useful model than standard Brownian motion.
- Examples (combining with the Donsker principle):
  - The amount won or lost in a game of chance that is not fair (approximating discrete winnings / losses with continuous changes).
  - The score difference between two competing sports teams (approximating this difference with a continuous function).
- This is a Gaussian process with continuous paths and stationary and independent increments.
- Example: Computing the chance of winning team game based on intermdiate score.
- Note: If a Brownian motion with drift is observed at points $y_1, \ldots, y_n$ and μ and σ are not fixed, there are priors so that we can do conjugate analysis, and analytically get a posterior process. However this posterior process is not a Gaussian process.

Geometric Brownian motion

- The stochastic process

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  where $G_0 > 0$ is called *geometric Brownian motion* with drift parameter μ and variance parameter σ^2.
- $\log(G_t)$ is a Gaussian process with expectation $\log(G_0) + \mu t$ and variance $t\sigma^2$.
- Show that
  - $\mathrm{E}(G_t) = G_0 e^{t(\mu+\sigma^2/2)}$
  - $\mathrm{Var}(G_t) = G_0^2 e^{2t(\mu+\sigma^2/2)}(e^{t\sigma^2} - 1)$
- Natural model for things that develop by multiplication of random independent factors, rather than addition of random independent increments. Example: Stock prices.

Modelling stock price with geometric Brownian motion

- To model the price of a stock, it is reasonable to
  - use a continuous-time stochastic model.
  - consider the *factor* with which it changes, not the differences in prices.
  - consider normal distributions for such factors (? at least for short time differences?)
  - use a parameter for the trend of the price, and one for the variability of the price.
  - make a Markov assumption(??? or not???)
- This leads to using a geometric Brownian motion as model

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  In this context σ is called the *volatility* of the stock.
- Example: A stock price is modelled with $G_0 = 67.3$, $\mu = 0.08$, $\sigma = 0.3$. What is the probability that the price is above 100 after 3 years?