MVE550 2023 Lecture 18
Dobrow Chapter 8, part 3

Petter Mostad

Chalmers University

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Review: Brownian motion

- A process $\{B_t\}_{t\geq 0}$ where $B_t \sim \text{Normal}(0, t)$. No parameters.
- Restarting Brownian motions at stopping times.
- First hitting times.
- Maximum of Brownian motion.
- Zeros of Brownian Motion.
- Brownian bridge.
- Brownian motion with a drift.

Example: Counting zeros

How many zeros does B_t have in the interval $[0, s]$?

- If L_s is the last zero in $[0, s)$, we have

$$L_s/s \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right),$$

  which means that $0 < L_s < s$ with probability 1.
- Repeating the argument, we have that $0 < L_{L_s} < L_s$ with probability 1.
- Conclusion: With probability 1, there is an infinite number of zeros $[0, s)$ for any $s > 0$.

Geometric Brownian motion

- The stochastic process

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  where $G_0 > 0$ is called *geometric Brownian motion* with drift parameter μ and variance parameter σ^2.
- $\log(G_t)$ is a Gaussian process with expectation $\log(G_0) + \mu t$ and variance $t\sigma^2$.
- Show that
  - $\mathrm{E}(G_t) = G_0 e^{t(\mu+\sigma^2/2)}$
  - $\mathrm{Var}(G_t) = G_0^2 e^{2t(\mu+\sigma^2/2)}(e^{t\sigma^2} - 1)$
- Natural model for things that develop by multiplication of random independent factors, rather than addition of random independent increments. Example: Stock prices.

Modelling stock price with geometric Brownian motion

- To model the price of a stock, it is reasonable to
  - use a continuous-time stochastic model.
  - consider the *factor* with which it changes, not the differences in prices.
  - consider normal distributions for such factors (? at least for short time differences?)
  - use a parameter for the trend of the price, and one for the variability of the price.
  - make a Markov assumption(??? or not???)
- This leads to using a geometric Brownian motion as model

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  In this context σ is called the *volatility* of the stock.
- Example: A stock price is modelled with $G_0 = 67.3$, $\mu = 0.08$, $\sigma = 0.3$. What is the probability that the price is above 100 after 3 years?

Stock options

- A (European) stock option is a right (but not obligation) to buy a stock at a given time t in the future for a given price K.
- How much can you expect to earn from a stock option at that future time?
- We get that (see next page)

$$\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 e^{t(\mu+\sigma^2/2)} \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

  where $\beta = (\log(K/G_0) - \mu t)/\sigma$.
- Example: A stock price is modelled with $G_0 = 67.3$, $\mu = 0.08$, $\sigma = 0.3$. What is the expected payoff from an option to buy the stock at 100 in 3 years?

Proof

- Prove the algebraic identity

$$e^{\sigma x}\,\mathrm{Normal}(x;0,t) = e^{\sigma^2 t/2}\,\mathrm{Normal}(x;\sigma t,t)$$

- Then, defining $\beta = (\log(K/G_0) - \mu t)/\sigma$, we get

$$\begin{aligned}
& \mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = \mathrm{E}\left(\max\left(G_0 e^{\mu+\sigma B_t} - K, 0\right)\right) \\
= & \int_{-\infty}^{\infty} \max\left(G_0 e^{\mu t+\sigma x} - K, 0\right) \mathrm{Normal}(x;0,t)\,dx \\
= & \int_{\beta}^{\infty} \left(G_0 e^{\mu t+\sigma x} - K\right) \mathrm{Normal}(x;0,t)\,dx \\
= & \; G_0 e^{\mu t} \int_{\beta}^{\infty} e^{\sigma x}\,\mathrm{Normal}(x;0,t)\,dx - K \int_{\beta}^{\infty} \mathrm{Normal}(x;0,t)\,dx \\
= & \; G_0 e^{t(\mu+\sigma^2/2)} \int_{\beta}^{\infty} \mathrm{Normal}(x;\sigma t,t)\,dx - K \int_{\beta}^{\infty} \mathrm{Normal}(x;0,t)\,dx \\
= & \; G_0 e^{t(\mu+\sigma^2/2)}\,\Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - K\,\Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)
\end{aligned}$$

Martingales

- A stochastic process $(Y_t)_{t\geq 0}$ is a *martingale* if for $t \geq 0$
  - $\mathrm{E}(Y_t \mid Y_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) < \infty$.
- Brownian motion is a martingale.
- $(Y_t)_{t\geq 0}$ is a *martingale with respect to* $(X_t)_{t\geq 0}$ if for all $t \geq 0$
  - $\mathrm{E}(Y_t \mid X_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) < \infty$.
- Example: Define $Y_t = B_t^2 - t$ for $t \geq 0$. Then Y_t is a martingale with respect to Brownian motion.

Geometric Brownian motion can be a martingale

Let $G_t = G_0 e^{\mu t + \sigma B_t}$ be Geometric Brownian motion. We get

$$
\begin{aligned}
& \mathrm{E}\left(G_t \mid B_r, 0 \le r \le s\right) \\
= \; & \mathrm{E}\left(G_0 e^{\mu t + \sigma B_t} \mid B_r, 0 \le r \le s\right) \\
= \; & \mathrm{E}\left(G_0 e^{\mu (t-s) + \sigma (B_t - B_s)} e^{\mu s + \sigma B_s} \mid B_r, 0 \le r \le s\right) \\
= \; & \mathrm{E}\left(G_{t-s}\right) e^{\mu s + \sigma B_s} \\
= \; & G_0 e^{(t-s)(\mu + \sigma^2/2)} e^{\mu s + \sigma B_s} \\
= \; & G_s e^{(t-s)(\mu + \sigma^2/2)}
\end{aligned}
$$

- We see that G_t is a martingale with respect to B_t if and only if $\mu + \sigma^2/2 = 0$.

Discounting future values of stocks

- When making investments, there is always a range of choices, some of which are sometimes called "risk free". Such investments may pay a fixed interest.
- When interests are compounded frequently, a reasonable model is that an investment of G_0 has a value $G_0 e^{rt}$ after time t, where r is the "risk free" investment rate of return.
- A common way to take this alternative into account is to instead "discount" all other investments with the factor e^{-rt}.
- For example, the *discounted* value of a stock can be modelled as

$$e^{-rt}G_t = e^{-rt}G_0 e^{\mu t + \sigma B_t} = G_0 e^{(\mu - r)t + \sigma B_t}$$

- A possible assumption about the trend μ of a stock price: The *discounted* value behaves as a Martingale with respect to Brownian motion.
- We get

$$\mu - r + \sigma^2/2 = 0, \qquad \text{i.e.,} \qquad \mu = r - \sigma^2/2.$$

The Black-Scholes formula for option pricing

- The Black-Scholes formula for option pricing is based on
  - Assuming the discounted stock price is a Martingale w.r.t. B_t.
  - Using discounting when computing the value of the option.
- We get

$$e^{-rt}\,\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - e^{-rt} K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

  where $\beta = (\log(K/G_0) - (r - \sigma^2/2)t)/\sigma$.
- Example: With $r = 0.02$, $G_0 = 67.3$, $\sigma = 0.3$, $t = 3$, and $K = 70$, we get the discounted stock option price 14.3534.