Exercise number	Homework	Class exercise	Possible exam type question
1			
2	✓		✓
3		✓	
4	✓		
5			
6	✓		
7		✓	✓
8			✓
9	✓		✓
10		✓	
11			✓
12		✓	✓
13	✓		
14		✓	
15		✓	✓
16	✓		✓
17		✓	✓
18			✓
19			
20			
21		✓	
22		✓	✓
23	✓		
24			✓
25		✓	✓
26			✓

Group Theory exercises

Here is a list of exercises on group theory. Those used as homework are without solution. All others have a solution attached.

Important! Those with a check-mark in the last column are similar to the type of questions that may show up in the exam.

EXERCISE 1.

Show that $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ w/ $\oplus$

$a \oplus b = a + b \mod n$

is an ABELIAN GROUP.

Find the SUBGROUPS of $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$.

Closure: $a + b \mod n \in \{0, 1 \dots, n-1\}$

Associative:

$(a \oplus b) \oplus c$: | $a \oplus b = q$ where $a + b = pn + q$ and $0 \leq q \leq n-1$ by def.

$a + b = pn + q$

$q + c = p'n + q'$ (also)

$\Rightarrow q' = a + b + c - (p + p')n$

$\equiv a + b + c \mod n.$

$a \oplus (b \oplus c)$: $b + c = p''n + q''$

$q'' + a = p'''n + q'''$

$\Rightarrow q''' = a + b + c - (p'' + p''')n$

$= (a + b + c) \mod n.$

Identity: $0 \oplus a = a$ obvious.

Inverse: $a \oplus (n-a) = a+n-a \mod n = 0$

Abelian: $a \oplus b = a+b \mod n =$
$= b+a \mod n = b \oplus a$.

To find subgroups, let's start w/ $\{0\}$ ALWAYS A SUBGROUP.

Try adding one element "a", note that for it to be a group all "multiples" of a must also be included.

$\mathbb{Z}_3$: $\{0\}$ ok.

try : $\{0,1\} \xrightarrow{\text{BUT}} 1 \oplus 1 = 2$ must also be included

$\Rightarrow$ only the full $\{0,1,2\}$

True for any $\mathbb{Z}_p$ ~~p~~ prime.

only $\{0\}$ and $\mathbb{Z}_p$ are subgroups.

$\mathbb{Z}_4$ has : $\{0\}$, $\{0,2\}$, $\{0,1,2,3\}$

$\mathbb{Z}_6$ has $\{0\}$, $\{0,2,4\}$, $\{0,3\}$.

EXERCISE 2

Find the multiplication table for a group w/ three elements and show it is unique.

EXERCISE 3

Show that the defining repr. of S_n is REDUCIBLE.

The def rep. of S_n consists of $n \times n$ matrices describing the permutation of n objects denoted by

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

e.g. the permutation switching $e_1 \leftrightarrow e_n$ is

$$\begin{pmatrix} 0 & 0 & \dots & & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

All matrices consist of "0" and "1" arranged so that there is only one "1" on each line / column.

The matrix $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & & \\ 1 & \dots & & & 1 \end{pmatrix}$ commutes with all of them and it's NOT $\propto \mathbb{1}$.

By Shur's lemma the repr. is REDUCIBLE.

EXERCISE 4

Find all groups with four elements and show they are all abelian.

EXERCISE 5

Suppose D_1 and D_2 are equivalent IRREPS : $D_2(g) = S D_1(g) S^{-1} \quad \forall g$.

Suppose $\exists A$ such that:

$\forall g \quad A D_1(g) = D_2(g) A$.

Find the general expression for A.

---

write $A D_1 = D_2 A$ as

$$A D_1 = S D_1 S^{-1} A$$

$$\Rightarrow S^{-1} A D_1 = D_1 S^{-1} A.$$

Shur : $S^{-1} A = \lambda \mathbb{1}$

$$\Rightarrow A = \lambda \cdot S$$

EXERCISE 6

Assume $[A,B] = B$ and compute

$$e^{i\alpha A} B e^{-i\alpha A}.$$

EXERCISE 7

For $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ compute $e^{i\alpha A}$.

One could always diagonalize, but here it is easier to notice:

$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A$

$A^4 = A^2$, $A^5 = A^3$ ... $\quad A^{\text{even}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A^{\text{odd}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

$$\Rightarrow e^{i\alpha A} = \cos\alpha A + i \sin\alpha A$$

$$\begin{aligned}
\cos\alpha A &= \mathbb{1} - \frac{1}{2}\alpha^2 A^2 + \frac{1}{4!}\alpha^4 A^4 + \dots \\
&= \mathbb{1} - \frac{1}{2}\alpha^2 A_{\text{even}} + \frac{1}{4!}\alpha^4 A_{\text{even}} + \dots \\
&= \mathbb{1} - A_{\text{even}} + A_{\text{even}} - \frac{1}{2}\alpha^2 A_{\text{even}} + \frac{1}{4!}\alpha^4 A_{\text{even}} + \dots \\
&= \mathbb{1} - A_{\text{even}} + A_{\text{even}}\left(1 - \frac{1}{2}\alpha^2 + \frac{1}{4!}\alpha^4 + \dots\right) \\
&= \mathbb{1} - A_{\text{even}} + A_{\text{even}} \cos\alpha = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix}
\end{aligned}$$

$$\sin \alpha A = \alpha A - \frac{1}{3!}\alpha^3 A^3 + \frac{1}{5!}\alpha^5 A^5 + \dots$$

$$= \alpha A_{odd} - \frac{1}{3!}\alpha^3 A_{odd} + \frac{1}{5!}\alpha^5 A_{odd}$$

$$= A_{odd} \cdot \sin\alpha = \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ \sin\alpha & 0 & 0 \end{pmatrix}$$

$$\Rightarrow e^{i\alpha A} = \begin{pmatrix} \cos\alpha & 0 & i\sin\alpha \\ 0 & 1 & 0 \\ i\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

sanity check: $\alpha = 0 \Rightarrow e^{i\alpha A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

EXERCISE 8

For $A = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ compute e^{7A}

Diagonalize A: $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda = 1, 2$

Eigenvectors : $\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow e^{7A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^7 & 0 \\ 0 & e^{14} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$

$= \frac{1}{2}\begin{pmatrix} e^7 + e^{14} & e^7 - e^{14} \\ e^7 - e^{14} & e^7 + e^{14} \end{pmatrix}$

EXERCISE 9

For $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ compute $\sin A$

EXERCISE 10

Given $g_1 = e^{i\alpha^a X^a}$ and $g_2 = e^{i\beta^b X^b}$

Construct $g_1 g_2$ to <u>third</u> order in the parameters α^a and β^b.

Set $g_1 g_2 = e^{i\delta^c X^c}$

Let $\delta^c = \delta^c_0 + \delta^c_1 + \delta^c_2 + \delta^c_3 + \dots$

where δ^c_n is a homogeneous polynomial in α^a and β^b of degree n.

$$g_1 g_2 = \left(1 + i\alpha\cdot X - \frac{1}{2}\alpha\cdot X\,\alpha\cdot X - \frac{i}{6}\alpha\cdot X\,\alpha\cdot X\,\alpha\cdot X + \dots\right)\times$$
$$\times\left(1 + i\beta\cdot X - \frac{1}{2}\beta\cdot X\,\beta\cdot X - \frac{i}{6}\beta\cdot X\,\beta\cdot X\,\beta\cdot X + \dots\right)$$

$$= 1 + i\alpha\cdot X + i\beta\cdot X - \frac{1}{2}\alpha\cdot X\,\alpha\cdot X$$
$$- \alpha\cdot X\,\beta\cdot X - \frac{1}{2}\beta\cdot X\,\beta\cdot X - \frac{i}{6}\alpha\cdot X\,\alpha\cdot X\,\alpha\cdot X$$
$$- \frac{i}{2}\alpha\cdot X\,\alpha\cdot X\,\beta\cdot X - \frac{i}{2}\alpha\cdot X\,\beta\cdot X\,\beta\cdot X - \frac{i}{6}\beta X\,\beta X\,\beta X$$
$$+ \text{higher orders}$$

To zeroth order $\delta_0^c = 0$.

To first order:

$$\mathbb{1} + i\alpha^a X^a + i\beta^b X^b = \mathbb{1} + i\delta_1^c X^c$$

$$\Rightarrow \delta_1^c = \alpha^c + \beta^c$$

to second order:

$$-\frac{1}{2}\alpha^a X^a \alpha^b X^b - \alpha^a X^a \beta^b X^b - \frac{1}{2}\beta^a X^a \beta^b X^b =$$

$$= i\delta_2^c X^c - \frac{1}{2}\delta_1^a X^a \delta_1^b X^b$$

$$\Rightarrow -\frac{1}{2}\alpha^a X^a \alpha^b X^b - \alpha^a X^a \beta^b X^b - \frac{1}{2}\beta^a X^a \beta^b X^b =$$

$$= i\delta_2^c X^c - \frac{1}{2}(\alpha^a + \beta^a) X^a (\alpha^b + \beta^b) X^b$$

$$\Rightarrow -\frac{1}{2}\cancel{\alpha^a X^a \alpha^b X^b} - \alpha^a X^a \beta^b X^b - \frac{1}{2}\cancel{\beta^a X^a \beta^b} X^b$$

$$= i\delta_2^c X^c - \cancel{\frac{1}{2}\alpha^a X^a \alpha^b} X^b - \frac{1}{2}\beta^a X^a \alpha^b X^b - \frac{1}{2}\alpha^a X^a \beta^b X^b$$

$$-\cancel{\frac{1}{2}\beta^a X^a \beta^b} X^b$$

$$\Rightarrow i\delta_2^c X^c = \frac{1}{2}\beta^a X^a \alpha^b X^b - \frac{1}{2}\alpha^a X^a \beta^b X^b$$

$$= \frac{1}{2}\beta^a \alpha^b [X^a X^b] = \frac{1}{2}\beta^a \alpha^b i f^{abc} X^c$$

$$\Rightarrow \delta_2^c = \frac{1}{2} f^{abc} \beta^a \alpha^b .$$

To third order:

$$-\frac{i}{6}\alpha^a X^a \alpha^b X^b \alpha^c X^c - \frac{i}{2}\alpha^a X^a \alpha^b X^b \beta^c X^c - \frac{i}{2}\alpha^a X^a \beta^b X^b \beta^c X^c$$

$$-\frac{i}{6}\beta^a X^a \beta^b X^b \beta^c X^c = i\,\delta_3^c X^c - \frac{1}{2}\times 2\, \delta_2^a X^a \delta_1^b X^b$$

$$-\frac{i}{6}\delta_1^a X^a \delta_1^b X^b \delta_1^c X^c = i\,\delta_3^c X^c$$

$$-\frac{1}{2} f^{dca} \beta^d \alpha^c X^a (\alpha^b + \beta^b) X^b -$$

$$-\frac{i}{6}(\alpha^a + \beta^a) X^a (\alpha^b + \beta^b) X^b (\alpha^c + \beta^c) X^c =$$

$$= i\,\delta_3^c X^c - \frac{1}{2} f^{dca} \beta^d \alpha^c (\alpha^b + \beta^b) X^a X^b -$$

$$-\frac{i}{6}\big(\alpha^a X^a \alpha^b X^b \alpha^c X^c + \alpha X \alpha X \beta X + \alpha X \beta X \alpha X$$

$$+ \beta X \alpha X \alpha X + \alpha X \beta X \beta X + \beta X \alpha X \beta X$$

$$+ \beta X \beta X \alpha X + \beta X \beta X \beta X\big)$$

$$i\delta_3^c X^c = \frac{1}{2} f^{dca}\beta^d\alpha^c(\alpha^b+\beta^b)X^aX^b$$

$$-\frac{i}{3}\alpha^aX^a\alpha^bX^b\beta^cX^c + \frac{i}{6}\alpha^aX^a\beta^bX^b\alpha^cX^c +$$

$$+\frac{i}{6}\beta^aX^a\alpha^bX^b\alpha^cX^c - \frac{i}{3}\alpha^aX^a\beta^bX^b\beta^cX^c$$

$$+\frac{i}{6}\beta^aX^a\alpha^bX^b\beta^cX^c + \frac{i}{6}\beta^aX^a\beta^bX^b\alpha^cX^c =$$

$$= \frac{1}{2} f^{dca}\beta^d\alpha^c(\alpha^b+\beta^b)X^aX^b - \frac{i}{3}\alpha^aX^a\alpha^b\beta^c[X^bX^c]$$

$$-\frac{i}{6}\alpha^aX^a\beta^bX^b\alpha^cX^c + \frac{i}{6}\beta^aX^a\alpha^bX^b\alpha^cX^c$$

$$-\frac{i}{3}\alpha^a\beta^b[X^aX^b]\beta^cX^c - \frac{i}{6}\beta^aX^a\alpha^bX^b\beta^cX^c +$$

$$+\frac{i}{6}\beta^aX^a\beta^bX^b\alpha^cX^c =$$

$$= \frac{1}{2} f^{dca}\beta^d\alpha^c(\alpha^b+\beta^b)X^aX^b - \frac{i}{3}\alpha^aX^a\alpha^b\beta^c i f^{bcd}X^d$$

$$+\frac{i}{6}\beta^a\alpha^b i f^{abd}X^d\alpha^cX^c - \frac{i}{3}\alpha^a\beta^b i f^{abd}X^d\beta^cX^c$$

$$+\frac{i}{6}\beta^aX^a\beta^b\alpha^c i f^{bcd}X^d =$$

(lots of index changes ...)

$$= \frac{1}{2} \underline{f^{cda} \beta^c \alpha^d X^a \alpha^b X^b} + \frac{1}{2} f^{cda} \beta^c \alpha^d \underline{X^a \beta^b} X^b$$

$$+ \underline{\frac{1}{3} \alpha^a X^a f^{cdb} \alpha^c \beta^d X^b} - \underline{\frac{1}{6} f^{cda} \beta^c \alpha^d X^a \alpha^b X^b}$$

$$+ \frac{1}{3} \underline{\underline{f^{cda} \alpha^c \beta^d X^a \beta^b X^b}} - \underline{\frac{1}{6} \beta^a X^a f^{cdb} \beta^c \alpha^d X^b} =$$

$$= \frac{1}{3} \alpha^a X^a f^{cdb} \alpha^c \beta^d X^b + \frac{1}{3} f^{cda} \beta^c \alpha^d X^a \alpha^b X^b$$

$$+ \frac{1}{6} f^{cda} \alpha^c \beta^d X^a \beta^b X^b - \frac{1}{6} \beta^a X^a f^{cdb} \beta^c \alpha^d X^b$$

$$= \frac{1}{3} \alpha^a f^{cdb} \alpha^c \beta^d \, i f^{abe} X^e + \frac{1}{6} f^{cda} \alpha^c \beta^d \beta^b \, i f^{abe} X^e$$

$$\Rightarrow \delta_3^e = \frac{1}{3} f^{cdb} f^{abe} \alpha^a \alpha^c \beta^d + \frac{1}{6} f^{cda} f^{abe} \alpha^c \beta^d \beta^b$$

EXERCISE 11

Compute $e^{i\vec{r}\vec{\sigma}}$ where $\vec{\sigma}$ are the Pauli matrices.

Write $\vec{r} = r\hat{r}$ where $\hat{r}\cdot\hat{r} = 1$.

$$(\hat{r}\cdot\vec{\sigma})^2 = \hat{r}^a\hat{r}^b\sigma^a\sigma^b = \frac{1}{2}\hat{r}^a\hat{r}^b\{\sigma^a\sigma^b\}$$

$$= \frac{1}{2}\hat{r}^a\hat{r}^b\cdot 2\,\delta^{ab}\cdot\mathbb{1}_{2\times 2} = \mathbb{1}$$

So: $$e^{i\vec{r}\vec{\sigma}} = \cos(r(\hat{r}\vec{\sigma})) + i\sin(r(\hat{r}\sigma))$$

$$\cos r(\hat{r}\vec{\sigma}) = \mathbb{1} - \frac{1}{2}r^2(\hat{r}\vec{\sigma})^2 + \frac{1}{4!}r^4(\hat{r}\vec{\sigma})^4 + \dots$$

$$= \mathbb{1} - \frac{1}{2}r^2\mathbb{1} + \frac{1}{4}r^4\mathbb{1} = \cos r\cdot\mathbb{1}$$

$$\sin r(\hat{r}\vec{\sigma}) = r(\hat{r}\vec{\sigma}) - \frac{1}{3!}r^3(\hat{r}\vec{\sigma})^3 + \frac{1}{5!}r^5(\hat{r}\vec{\sigma})^5 + \dots$$

$$= (\hat{r}\vec{\sigma})\left(r - \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \dots\right) =$$

$$= \hat{r}\vec{\sigma}\sin r$$

$$\Rightarrow e^{i\vec{r}\vec{\sigma}} = \cos r\,\mathbb{1} + i\,\hat{r}\vec{\sigma}\sin r =$$

$$= \cos r\,\mathbb{1} + i\,\vec{r}\vec{\sigma}\,\frac{\sin r}{r}$$

EXERCISE 12

Show that the states in the product of a spin 7 with a spin 3 irrep are in one to one correspondence w/ the states in the sum of a spin $\underset{\overset{\|}{7-3}}{4}, 5, 6, 7, 8, 9, \underset{\overset{\|}{7+3}}{10}$

A spin 3 irrep consist of $|S, M\rangle =$

$$|3,3\rangle, |3,2\rangle, |3,1\rangle, |3,0\rangle$$
$$|3,-1\rangle, |3,-2\rangle, |3,-3\rangle \quad (7 \text{ states})$$

A spin 7 : $\underbrace{|7\,7\rangle, |7\,6\rangle, \ldots \quad |7,-7\rangle}_{15 \text{ states}}$

The product of the two consists of all the $7 \times 15 = 105$ states. Let us order them in order of decreasing $M_1 + M_2$:

$|33\rangle|77\rangle$

$|32\rangle|77\rangle\ ,\ |33\rangle|76\rangle$

$|31\rangle|77\rangle\ ,\ |32\rangle|76\rangle\ ,\ |33\rangle|75\rangle$

$\vdots$

$|3,-3\rangle|7,-6\rangle\quad |3,-2\rangle|7,-7\rangle$

$|3,-3\rangle|7,-7\rangle.$

21 rows.

Clearly $|33\rangle|77\rangle = ||10,10\rangle$.
Acting on $||10,10\rangle$ w J_- 21 times fills the spin 10 irrep. one linear comb. from each row disappears. The $\perp$ combination in the II row becomes $|9,9\rangle$. Repeating the process we fill in the spin 9.
The longest row is the one with $M_1+M_2=0$:

$|3\,-3\rangle|7\,3\rangle, |3\,-2\rangle|7\,2\rangle, |3\,-1\rangle|7\,1\rangle, |3\,0\rangle|7\,0\rangle$
$|3\,1\rangle|7\,-1\rangle, |3\,2\rangle|7\,-2\rangle, |3\,3\rangle|7\,-3\rangle$
containing 7 states, so I can repeat 7 times going from 10 to 3.

EXERCISE 13

Show that the spin = 1 irrep of SU(2) is the adjoint representation.
Find the similarity tranformation that maps the adjoint

$$(\tilde{J}^a)^b_c = -i f_{abc} = -i \epsilon_{abc}$$

to the standard

$$J^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

EXERCISE 14

Construct explicitly $D_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ of S_3

(Call $D_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = D$)

First construct one element of the space:

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = |123\rangle + |213\rangle - |321\rangle - |231\rangle = \epsilon_1$$

Then act on it with the elements of S_3:

$$\epsilon_1 \downarrow$$

$e: \quad |123\rangle + |213\rangle - |321\rangle - |231\rangle = \epsilon_1$

$(123): \quad |312\rangle + |321\rangle - |132\rangle - |123\rangle \overset{def}{=} \epsilon_2$

$(132): \quad |231\rangle + |132\rangle - |213\rangle - |312\rangle = -\epsilon_1 - \epsilon_2$

$(12): \quad |213\rangle + |123\rangle - |231\rangle - |321\rangle = +\epsilon_1$

$(13): \quad |321\rangle + |312\rangle - |123\rangle - |132\rangle = \epsilon_2$

$(23): \quad |132\rangle + |231\rangle - |312\rangle - |213\rangle = -\epsilon_1 - \epsilon_2$

Do the same for ϵ_2 :

$$\overset{\epsilon_2 \downarrow}{}$$

$$e: \quad |312\rangle + |321\rangle - |132\rangle - |123\rangle = \epsilon_2$$

$$(123): \quad |231\rangle + |132\rangle - |213\rangle - |312\rangle = -\epsilon_1 - \epsilon_2$$

$$(132): \quad |123\rangle + |213\rangle - |321\rangle - |231\rangle = \epsilon_1$$

$$(12): \quad |132\rangle + |231\rangle - |312\rangle - |213\rangle = -\epsilon_1 - \epsilon_2$$

$$(13): \quad |213\rangle + |123\rangle - |231\rangle - |321\rangle = \epsilon_1$$

$$(23): \quad |321\rangle + |312\rangle - |123\rangle - |132\rangle = \epsilon_2$$

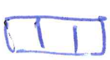
From which, writing $\epsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\epsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we read off:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(123) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad D(132) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$D(12) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad D(13) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad D(23) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

The repr as it stands is NOT UNITARY but I can already compute the characters:

$$\chi([\![e]\!]) = 2 \quad \chi([\![(123)]\!]) = -1, \quad \chi([\![(12)]\!]) = 0$$

  .

Let's unitarize it!

$$X^2 = \sum_g D(g)^\dagger D(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} +$$

$$+ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\Rightarrow X^2 = \Omega D^2 \Omega^T$$

where $\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}$

$$\Rightarrow X = \Omega D \Omega^T = \begin{pmatrix} 1+\sqrt{3} & -1+\sqrt{3} \\ -1+\sqrt{3} & 1+\sqrt{3} \end{pmatrix}$$

✓UNITARY

$$\hat{D}(e) = X D(e) X^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{D}((123)) = \quad '' \quad = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\hat{D}((132)) = \quad = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\hat{D}((12)) = \quad = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$$

$$\hat{D}((13)) = \quad = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{D}((23)) = \quad = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ -\frac{1}{2} & \sqrt{3}/2 \end{pmatrix}$$

Exercise 15

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Compute $tr(A \oplus B)$
$tr(A \otimes B)$
$\det(A \oplus B)$
$\det(A \otimes B)$

$tr(A \oplus B) = tr A + tr B = 0 + 3 = 3.$

$tr(A \otimes B) = tr A \cdot tr B = 0 \cdot 3 = 0$

$\det(A \oplus B) = \det A \cdot \det B = -1 \cdot 2 = -2$

$\det(A \otimes B) = (\det A)^2 \cdot (\det B)^2 = (-1)^2 \cdot 2^2 = 4.$

EXERCISE 16

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Compute $\exp(A \otimes B)$

Exercise 17

Decompose $D_{[2,1]} \otimes D_{[2,1]}$. (Use the character table in the notes!)

Let $D = D_{[2,1]} \otimes D_{[2,1]}$ of dim $= 2 \cdot 2 = 4$.

$$\chi_D(e) = \chi_{D_{[2,1]}}(e) \cdot \chi_{D_{[2,1]}}(e) = 2 \cdot 2 = 4$$

$$\chi_D(((12))) = \chi_{D_{[2,1]}}(((12))) \cdot \chi_{D_{[2,1]}}(((12))) = 0 \cdot 0 = 0$$

$$\chi_D(((123))) = \chi_{D_{[2,1]}}(((123))) \cdot \chi_{D_{[2,1]}}(((123))) = (-1) \cdot (-1) = 1$$

$$\chi_D(g) = \chi_{D_{[3]}}(g) + \chi_{D_{[2,1]}}(g) + \chi_{D_{[1,1,1]}}(g)$$

$$\Rightarrow \quad D_{[2,1]} \otimes D_{[2,1]} \sim D_{[3]} \oplus D_{[2,1]} \oplus D_{[1,1,1]}$$

EXERCISE 18.

Identify four subgroups of S_4 that are isomorphic to S_3.

S_4 permutes $\{1,2,3,4\}$. If I keep one fixed (i.e. w/ cycle (n)) the remaining permutations permute the 3 remaining elements, $\sim S_3$.

ex. keeping 4 fixed:

$$\left.\begin{array}{l} (1)(2)(3)(4) \\ (12)(3)(4) \\ (13)(2)(4) \\ (23)(1)(4) \\ (123)(4) \\ (132)(4) \end{array}\right\} \sim S_3 \text{ OK}$$

BUT NOT all the others:

$$\left.\begin{array}{l} \cancel{(12)(34)} \\ \cancel{(1)(2)(34)} \\ \vdots \\ \cancel{(124)(3)} \\ \vdots \end{array}\right\} \text{they "move" } 4_4.$$

EXERCISE 19

What is the order (i.e. # elements) of the subgroups of S_4 that:

i) leave 1 invariant

ii) leave 1 and 2 invariant

iii) leave the SET $\{1,2\}$ invariant.

---

i) The subgroup that leaves 1 invariant is S_3 = permuting 2, 3 and 4. It has ORDER = 6.

ii) To leave 1 AND 2 invariant I can only use the identity or switch 3 and 4:

$$e = (1)(2)(3)(4)$$
$$g = (1)(2)(34)$$

$\Rightarrow$ ORDER = 2

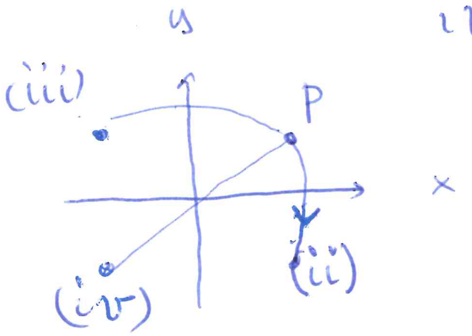
iii). In this case I can ALSO switch 1 and 2 but not exchange them with 3 or 4:

$$\underbrace{(1)(2)(3)(4),\ (12)(3)(4),\ (12)(34),\ (1)(2)(34)}_{\text{ORDER } = 4.}$$

EXERCISE 20

Consider $\mathbb{R}^2$ with the following operations:

i) Identity
ii) reflection along x
iii) " " y
iv) " in the origin:



(called Kleins fourgroup.)

- Construct the multiplication table
- Show it is abelian
- Find its subgroups

	i	ii	iii	iv
i	i	ii	iii	iv
ii	ii	i	iv	iii
iii	iii	iv	i	ii
iv	iv	iii	ii	i

The multiplication table is symmetric along the diagonal

$\Rightarrow$ ABELIAN.

The group is clearly isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\pm 1, \pm 1)$ with multiplic.$\}$

$\Rightarrow$ Subgroups: $\{e\} = \{(1,1)\}$.

$\{(\pm 1, 1)\} \simeq \mathbb{Z}_2$

$\{(1, \pm 1)\} \simeq \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2$

EXERCISE 21

Show that if $x^2 = e \quad \forall x \in G$
then G is ABELIAN.

Consider xy. $(xy)^2 = e$ by assumption

$\Rightarrow xyxy = e$

$\Rightarrow x^2yxy = x$

$\Rightarrow yxy = x$

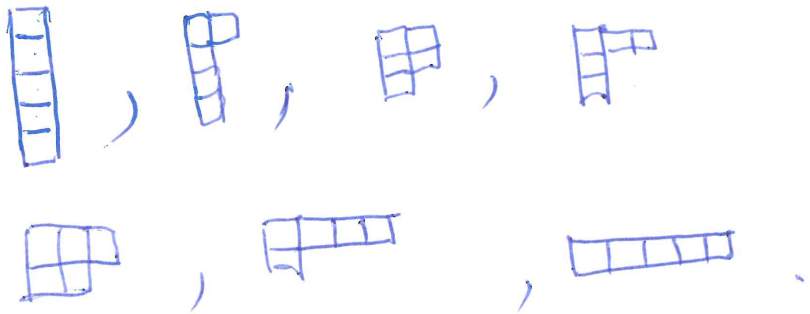
$\Rightarrow yxy^2 = xy$

$\Rightarrow yx = xy$ ✓

EXERCISE 22.

Find all conjugacy classes of S_5.

The c.c. are given by the different types of cycle lengths of a perm. Equivalently by the Young Tableaux:



EXERCISE 23

Show that S_4 has 5 irreps of dimension $1, 1, 2, 3, 3$.

EXERCISE 24

Consider the group with 3 elements $\{e, a, b\}$. Consider the matrices:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(a) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad D(b) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

1. Do they form a representation?
2. Is it faithful?
3. Is it unitary?
4. Is it irreducible?
5. Can it be used in the orthogonality theorem?

---

1. The only group with 3 elements has multiplications

$aa = b$, $bb = a$, $ab = ba = e$

One can easily check that D's obey the same relations

e.g. $D(a)D(a) = \frac{1}{4}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} =$

$= \frac{1}{4}\begin{pmatrix} 1-3 & -2\sqrt{3} \\ 2\sqrt{3} & -3+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = D(b)$

2. Yes : $g_1 \neq g_2 \Rightarrow D(g_1) \neq D(g_2)$

(all D's are different).

3. YES $D(e)^\dagger D(e) = D(a)^\dagger D(a) = D(b)^\dagger D(b) = \mathbb{1}$

(Check the "rows"^{2} = 1 and are $\perp$ to each other)

4. No It cannot be, since the group is abelian and all irreps of an abelian group are ONE-DIM.

5. No The theorem requires using irreps

EXERCISE 25

Let σ^a and τ^a $(a=1,2,3)$ be two sets of Pauli matrices acting on seperate copies of $\mathbb{C}^2$.
Let $e_1=\binom{1}{0}$ $e_2=\binom{0}{1}$ be a basis for the first $\mathbb{C}^2$ and $\epsilon_1=\binom{1}{0}$ $\epsilon_2=\binom{0}{1}$ a basis for the second.

Let $E_1=e_1\otimes\epsilon_1$, $E_2=e_1\otimes\epsilon_2$, $E_3=e_2\otimes\epsilon_1$
$E_4=e_2\otimes\epsilon_2$ be a basis in
$\mathbb{C}^4=\mathbb{C}^2\otimes\mathbb{C}^2$.

Construct the matrix $\sigma^2\otimes\tau^1=M$ and compute $\mathrm{tr}\,M$ and $\det M$.

Recall: $\sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\tau^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$

meaning $\sigma^2 e_1=ie_2$, $\sigma^2 e_2=-ie_1$
$\tau^1\epsilon_1=\epsilon_2$, $\tau^1\epsilon_2=\epsilon_1$

$$ME_1=(\sigma^2\otimes\tau^1)e_1\otimes\epsilon_1=\sigma^2e_1\otimes\tau^1\epsilon_1=ie_2\otimes\epsilon_2=iE_4$$

similarly:

$$ME_2=ie_2\otimes\epsilon_1=iE_3,\ ME_3=-E_2,\ ME_4=-E_1$$

$$\Rightarrow M = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{tr}\, M = 0$ $\quad$ $\det M = +1$

agrees $\quad$ with $\mathrm{tr}(\sigma^2 \otimes \tau^1) = \mathrm{tr}\,\sigma^2 \cdot \mathrm{tr}\,\tau^1 = 0$

$\det(\sigma^2 \otimes \tau^1) = (\det \sigma^2)^2 (\mathrm{tr}\,\tau^1)^2 = 1$

EXERCISE 26

Often in physics we do not write $\otimes$ expeicitly. Consider 2 sets of Pauli matrices acting on two seperate copies of $\mathbb{C}^2$.

Abbreviate
$$\sigma^a \otimes \tau^b = \sigma^a \tau^b$$
$$\sigma^a \otimes \mathbb{1} = \sigma^a$$
$$\mathbb{1} \otimes \tau^b = \tau^b$$
$$\mathbb{1} \otimes \mathbb{1} = \mathbb{1}$$

ANTI COMMUTATOR

Compute
a) $[\sigma^a, \sigma^b \tau^c]$
b) $\mathrm{tr}\left(\sigma^a \{\tau^b, \sigma^c \tau^d\}\right)$
c) $[\sigma^1 \tau^1, \sigma^2 \tau^2]$

---

a) $[\sigma^a, \sigma^b \tau^c] = \sigma^a \sigma^b \otimes \tau^c - \sigma^b \sigma^a \otimes \tau^c$
$= [\sigma^a, \sigma^b] \otimes \tau^c = 2i \in^{abd} \sigma^d \tau^c$

b) $\mathrm{tr}\left(\sigma^a \{\tau^b, \sigma^c \tau^d\}\right) = \mathrm{tr}\left(\sigma^a \left(\sigma^c \otimes \{\tau^b \tau^d\}\right)\right)$
$= \mathrm{tr}\left(\sigma^a \sigma^c \otimes \{\tau^b, \tau^d\}\right) = \mathrm{tr}(\sigma^a \sigma^c) \cdot \mathrm{tr}\{\tau^b \tau^d\}$
$= 2\delta^{ac} \cdot 4\delta^{bd} = 8\delta^{ac}\delta^{bd}$

c) $[\sigma^1\tau^1, \sigma^2\tau^2] = (\sigma^1\otimes\tau^1)(\sigma^2\otimes\tau^2) - (\sigma^2\otimes\tau^2)(\sigma^1\otimes\tau^1)$

$= \sigma^1\sigma^2\otimes\tau^1\tau^2 - \sigma^2\sigma^1\otimes\tau^2\tau^1 =$

$= i\sigma^3\otimes i\tau^3 - (-i\sigma^3)\otimes(-i\tau^3) =$

$= -\sigma^3\otimes\tau^3 + \sigma^3\otimes\tau^3 = 0$.