Special Relativity exercises

Here is a list of solutions to the problems in the compendium on special relativity. The relevant chapters for this course are 6, 7 and 8, although you can look at the other ones as well. The problems used as homework exercises are without solution.

Important! Those with a check-mark in the last column are similar to the type of questions that may show up in the exam.

Exercise number	Homework	Class exercise	Possible exam type question
6.1			✓
6.2	✓		✓
6.3			✓
6.4			✓
6.5	✓		✓
6.6			
6.7			
6.8			
7.1			
7.2			
7.3		✓	
7.4	✓		
7.5			
7.6			
7.7			
7.8	✓		✓
7.9			✓
7.10		✓	
7.11	✓		✓
7.12		✓	✓
7.13			
7.14	✓		
7.15			✓
7.16			
7.17			
7.18		✓	✓
7.19			
7.20			
7.21			
7.22		✓	
8.1	✓		✓
8.2			
8.3			
8.4			
8.5	✓		✓
8.6			
8.7			
8.8	✓		✓
8.9		✓	✓
8.10			
8.11		✓	✓
8.12			

2.1

Set $c=1$ for convenience.

$$x' = \gamma(x - vt) = \gamma\Big(\gamma(x' + vt') - v\big(\gamma(t' + vx')\big)\Big)$$

$$= \gamma^2 x' + \cancel{v\gamma t'} - \cancel{v\gamma t'} - \gamma^2 v^2 x'$$

$$= \gamma^2(1 - v^2)\, x' = x'$$

Similarly for all the others

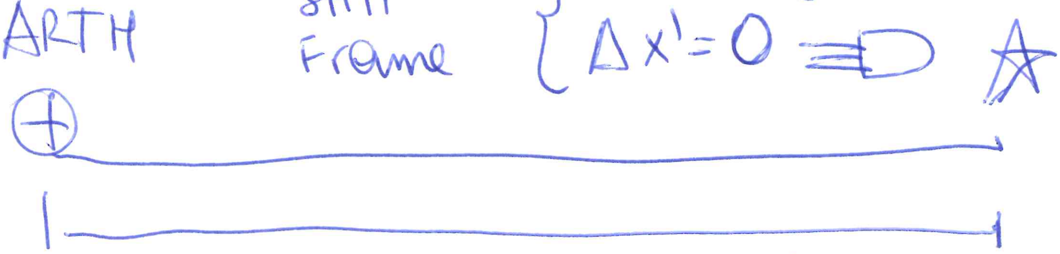
2.2 Units first:

1 yr = 1 year, unit of time

1 ly = 1 light-year = $c \times 1$ year, unit of length.

EARTH

In the SHIP frame $\begin{cases} \Delta t' = m\,\text{yr} \\ \Delta x' = 0 \end{cases} \Longrightarrow$ STAR



$\Delta x = n$ ly in the Earth frame.

a)
$$\Delta x = \gamma(v)\left(\Delta x' + v\Delta t'\right)$$
$$n\,\text{ly} = \gamma(v)\left(0 + v \cdot m\,\text{yr}\right)$$
$$n \cdot c\,\cancel{\text{yr}} = \gamma(v)\, v\, m \cdot \cancel{\text{yr}}$$
$$n c = \frac{v m}{\sqrt{1 - v^2/c^2}}$$

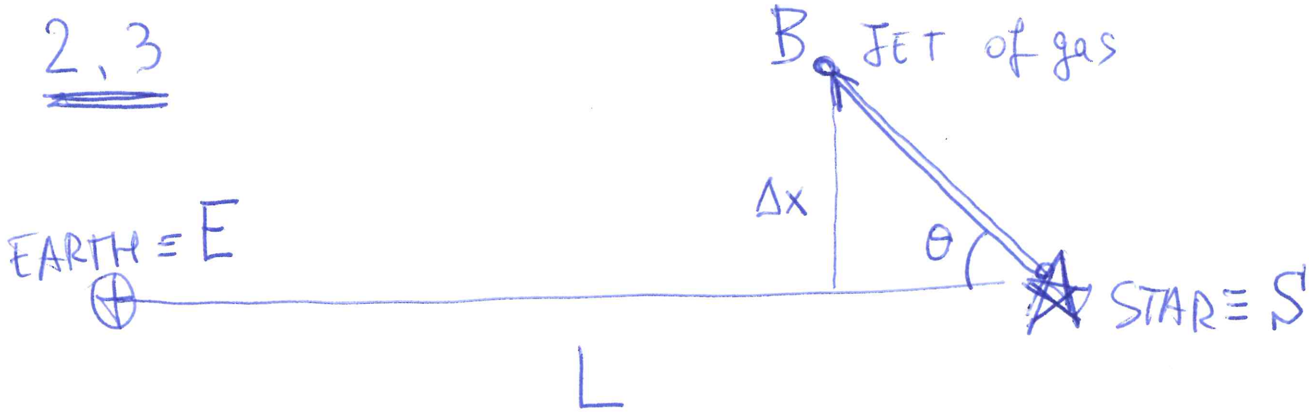
solve for v: $$v = \frac{n c}{\sqrt{n^2 + m^2}}$$

b) Note that $v < c$ always, so there are no limitations on n and m (as long as they are positive, obviously ...).

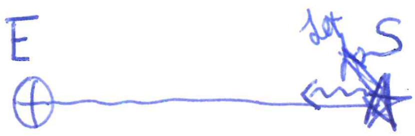
c) 4×10^{13} km $\cong 4{,}23$ ly

$$v = \frac{4{,}23}{\sqrt{4{,}23^2 + 10^2}} c = 0{,}389\, c$$

2,3



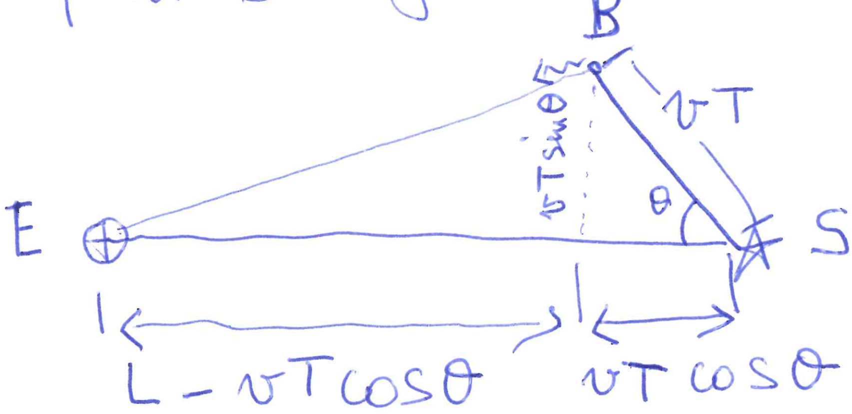
Suppose the jet is emitted at $t=0$ in the Earth frame:



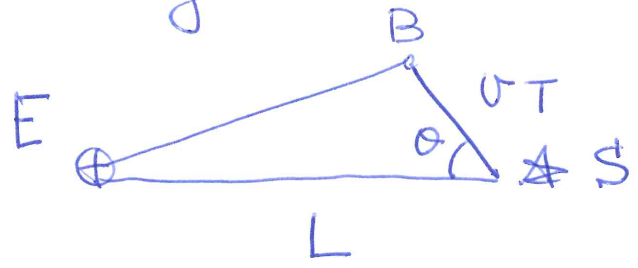
$t=0$ emission at point S

Light from the emission reaches the Earth after a time $t_1 = L/c$.

Suppose the jet reaches some point B after a time T



Using the cos- theorem:



$$EB = \sqrt{L^2 + v^2T^2 - 2LvT\cos\theta}$$

you can do the calculation exactly but you can make your life much easier by noticing that

$L \gg vT$ (The distance to the star is >> the lenght of the jet).

$$\Rightarrow EB \simeq L - vT\cos\theta$$

So, the time it takes for the light from B to reach the Earth:

$$t_2 \cong T + \frac{L - vT\cos\theta}{c}$$

Now if you just think (wrongly) at the jet as emitted $\perp$ The the ES, you estimate the transverse velocity as:

(a) $$V_{\perp}^{\text{apperent}} = \frac{\Delta x}{\Delta t} = \frac{vT\sin\theta}{t_2 - t_1} =$$

$$= \frac{vT\sin\theta}{T - \frac{v}{c}T\cos\theta} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}$$

that CAN BE $> c$ (!) for $v < c$:

(b) $$c \overset{\substack{\text{Limit case}\\ \downarrow}}{=} \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} \Rightarrow v = \frac{c}{\sin\theta + \cos\theta}$$

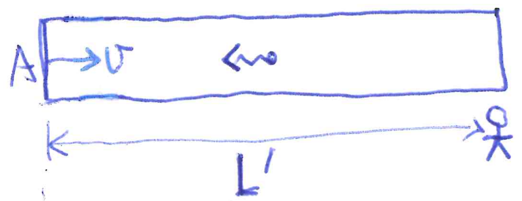
note that $\sin\theta + \cos\theta$ is always ≥ 1 for $\theta \in [0, \pi/2]$.

2.4 We always work in the ref. system of observer 2 (on the ground).

Let us look at light bouncing off A:

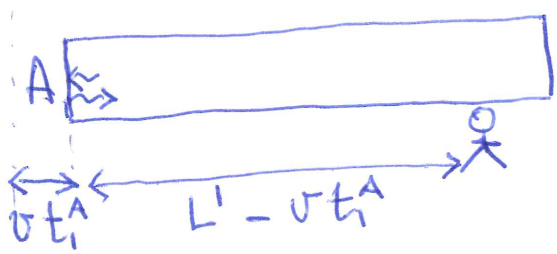
$(L' = \sqrt{1 - v^2/c^2}\, L)$

$t_0^A = 0$



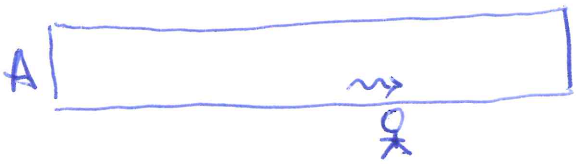
Light is emitted at the midpoint.

$$t_1^A = \frac{L'/2}{c+v}$$



Light bounces off the mirror A

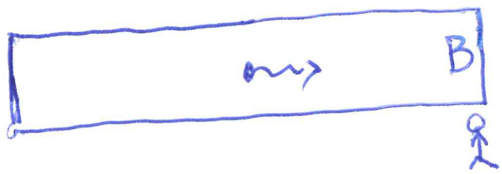
$$t_2^A = t_1^A + \frac{L' - v t_1^A}{c}$$



Light arrives at observer 2

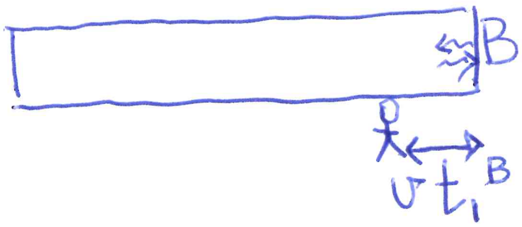
Similarly for light bouncing off B:

$t_0^B = 0$



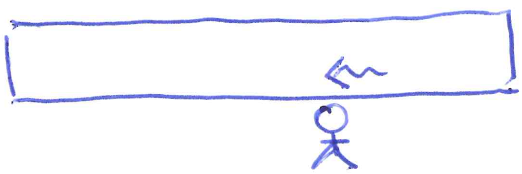
Light is emitted also at $t = 0$.

$$t_1^B = \frac{L'/2}{c-v}$$



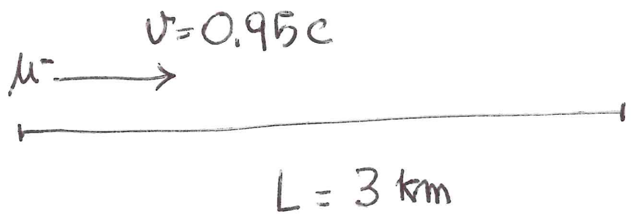
Light bounces off mirror B

$$t_2^B = t_1^B + \frac{v t_1^B}{c}$$



Light reaches observer 2

3.1



the time it takes in the lab is

$$t = L/v = \frac{\cancel{3} \times 10^3\,m}{0.95 \cdot \cancel{3} \cdot 10^8\,m/s} = 1.05 \times 10^{-5}\,s$$

In the μ^- frame $t_0 = \frac{t}{\gamma} =$

$$= 1.05 \times 10^{-5}\,s\,\sqrt{1 - 0.95^2} = 3.29 \cdot 10^{-6}\,s = 3.29\,\mu s$$

μ^- left: $N_0 \cdot e^{-t_0/\tau} = N_0\, e^{-\frac{3.29}{2.2}} = 0.224\, N_0$

22% μ^- survive.

(much more than without γ factor

$N_0\, e^{-t/\tau} \simeq 0.008\, N_0 \Rightarrow \simeq 0.8\%$ survive).

3.2 i) The lenght of the car in the garage frame is

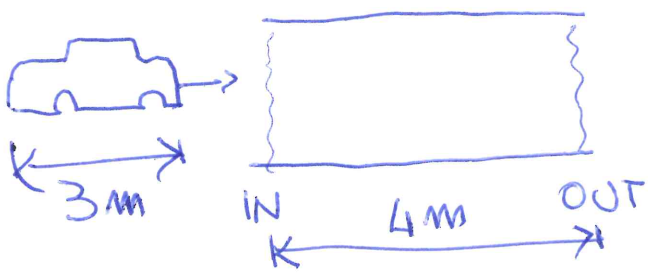
$$L'_{car} = 5 \cdot \sqrt{1-\left(\tfrac{4}{5}\right)^2}\ m = 3\,m$$

ii) The lenght of the garage in the cars frame is

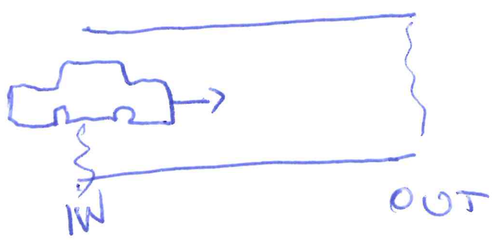
$$L'_{garage} = 4 \times \sqrt{1-\left(\tfrac{4}{5}\right)^2}\ m = 2.4\,m.$$

Here no collision occurs and the situation is easier to analyze:

a) GARAGE FRAME

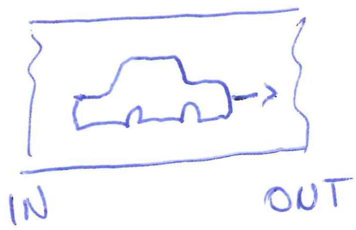


P_{IN}, P_{OUT} both OFF

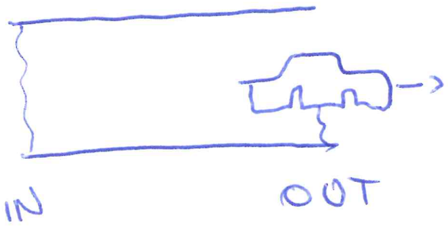


P_{IN} = ON

P_{OUT} = OFF

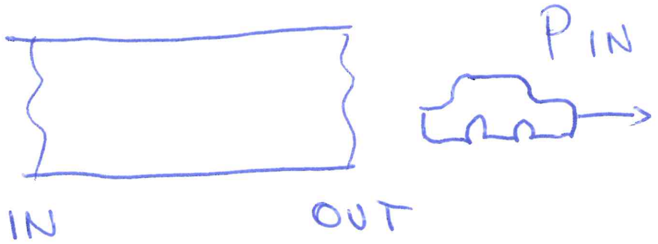


$P_{IN} = P_{OUT} = OFF$



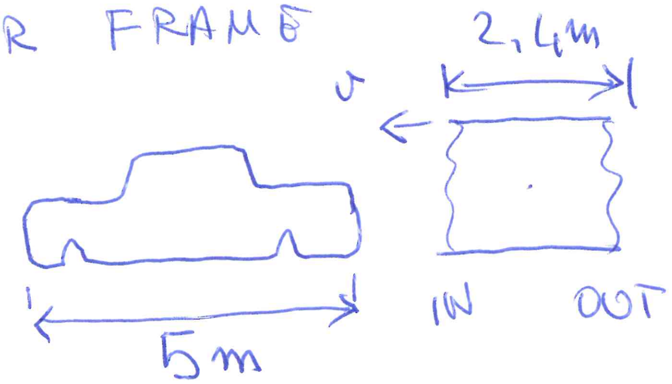
$P_{IN} = OFF$

$P_{OUT} = ON$

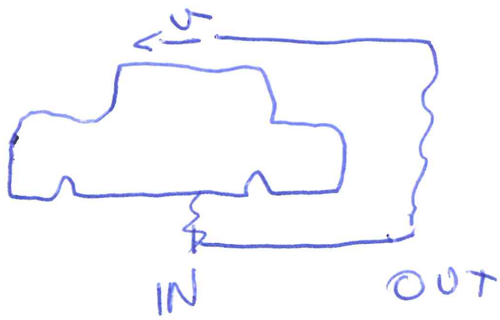


$P_{IN} = P_{OUT} = OFF$

(b) CAR FRAME

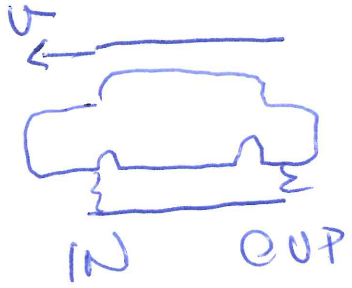


$P_{IN} = P_{OUT} = OFF$

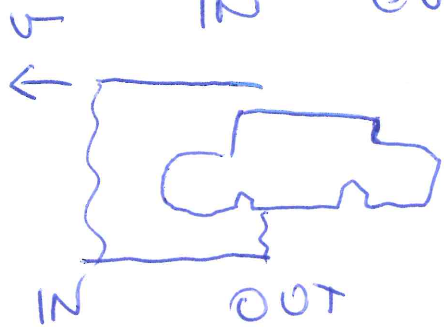


$P_{IN} = ON$

$P_{OUT} = OFF$

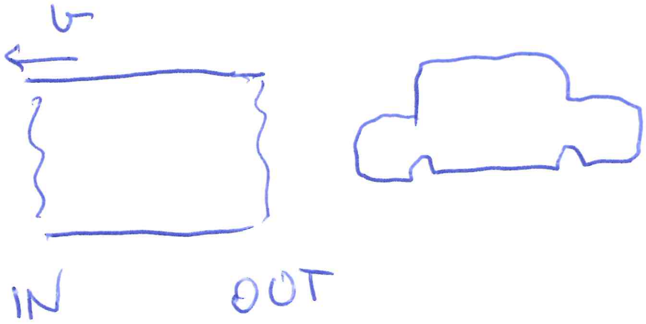


$P_{IN} = P_{OUT} = ON$



$P_{IN} = OFF$

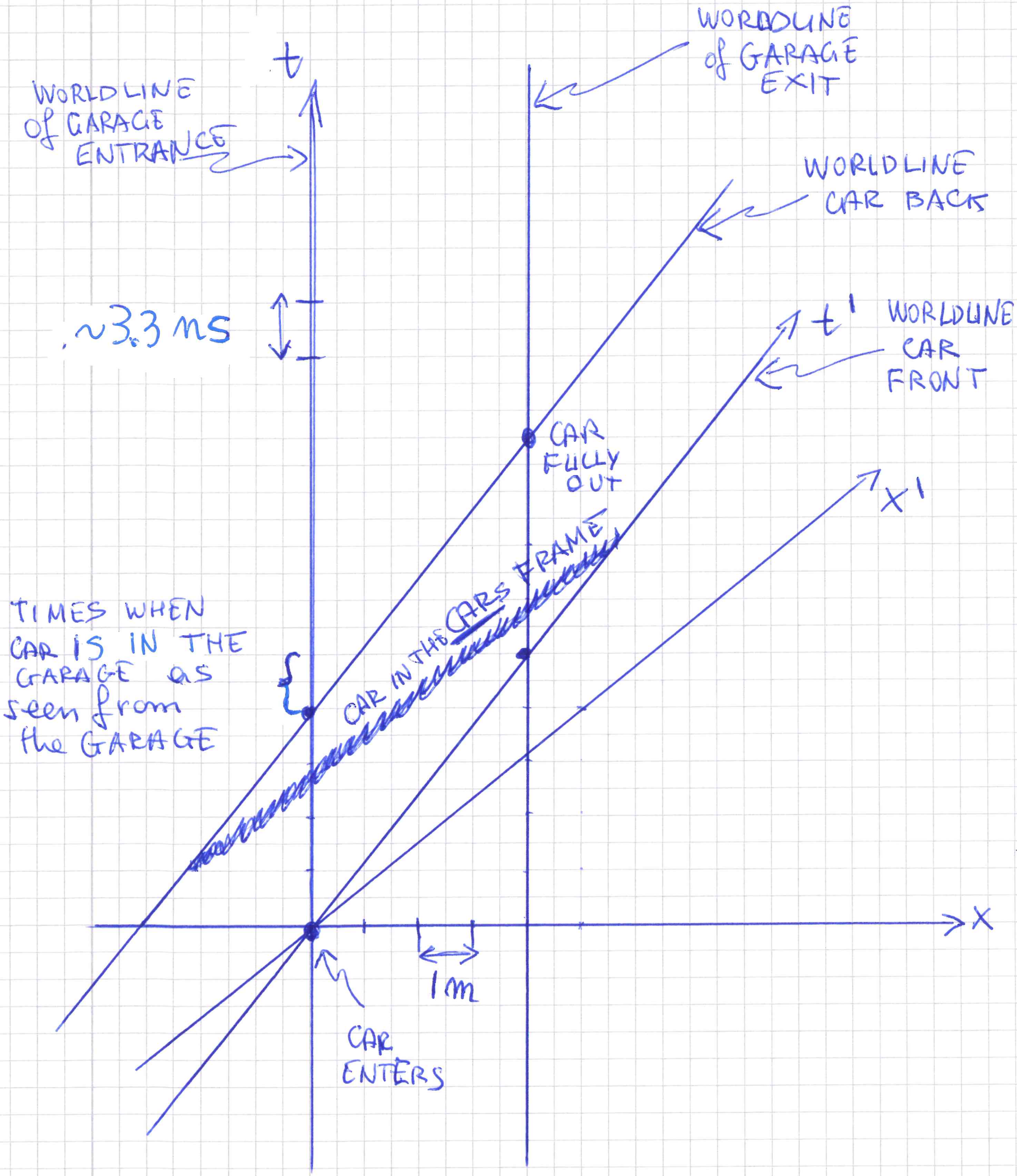
$P_{OUT} = ON$



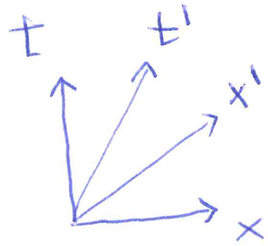
$P_{IN} = P_{OUT} = OFF$.

(c) No we cannot answer, It depends on the frame. The definition "in the garage" is as relative as the simultaneity of event at the two different places IN and OUT.

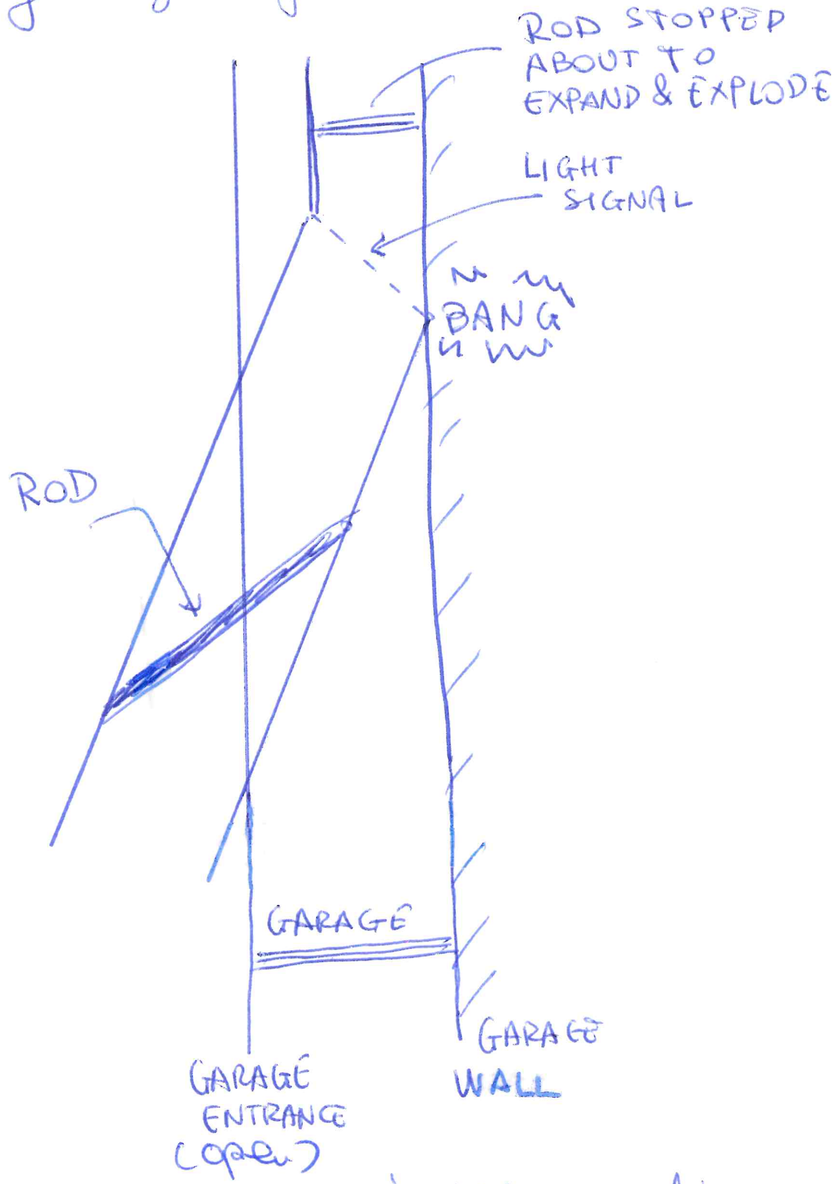
3.3


WORLDLINE of GARAGE EXIT
t
WORLDLINE of GARAGE ENTRANCE
WORLDLINE CAR BACK
t'
WORLDLINE CAR FRONT
~3.3 ms
CAR FULLY OUT
x'
CAR IN THE CARS FRAME
TIMES WHEN CAR IS IN THE GARAGE as seen from the GARAGE
x
1m
CAR ENTERS

COMPARE WITH THE ROD in the GARAGE of section 3.2



From the garage frame:



see Ladder Paradox in Wikipedia.

3.4

We can find the velocity of the ship in the same way as problem 2.2

$$\Delta x = \gamma(v)\left(\Delta x' + v\,\Delta t'\right)$$

$$10\,c = \gamma(v)\left(0 + 5 \cdot v\right)$$

$$\Rightarrow \quad v = \frac{10\,c}{\sqrt{10^2 + 5^2}} = 0{,}89\,c$$

The trip takes from Earth prespective:

$$T = 2 \cdot \frac{10 \times c \times yr}{0{,}89 \times c} = 22{,}4\ yr.$$

3,5

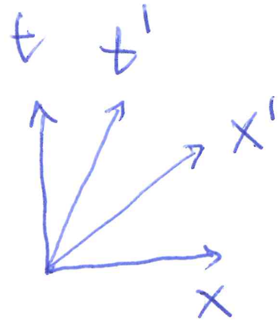
a) $T = \gamma \tau = \frac{1}{\sqrt{1 - v^2/c^2}} \times \tau =$

$= \frac{26\,\mathrm{ms}}{\sqrt{1 - 0{,}95^2}} = 83.3\,\mathrm{ms}$

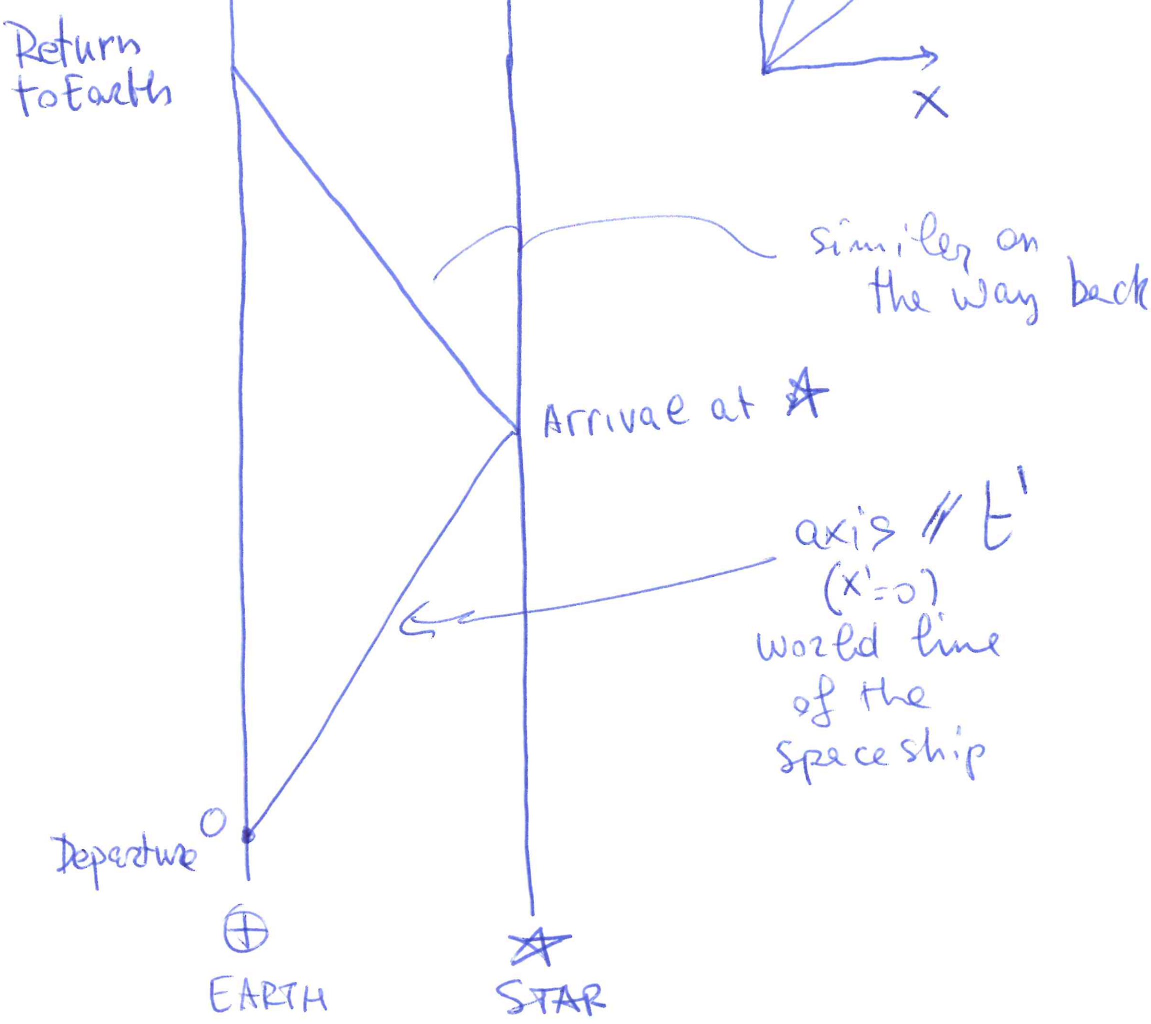
b) $d = vT = 0{,}95 \times \underbrace{\left(30\,\frac{\mathrm{cm}}{\mathrm{ns}}\right)}_{=c} \times 83.3\,\mathrm{ms}$

$= 23{,}7\,\mathrm{m}$

3.6



From Earth frame:



See Twin Paradox on Wikipedia.

4.1

a) $\begin{cases} x = A\omega t \\ y = A\sin\omega t \\ z = 0 \end{cases} \Rightarrow \begin{cases} v_x = A\omega \\ v_y = A\omega\cos\omega t \\ v_z = 0 \end{cases} \Rightarrow \begin{cases} a_x = 0 \\ a_y = -A\omega^2\sin\omega t \\ a_z = 0 \end{cases}$

b) $v^2 = A^2\omega^2(1+\cos^2\omega t)$ must be $< c^2$

but $1+\cos^2\omega t \in [1,2] \ \forall t$

$\Rightarrow 2A^2\omega^2 < c^2 \Rightarrow A\omega < c/\sqrt{2}$.

c) Use eqs (4.4) but careful that the boost in the problem is along the z-axis. It's probably easier to ① relabel $z \to x$, $x \to y$, $y \to z$

② Apply (4.4) as it is since the boost is along the new x

③ relabel back $x \to z$ $y \to x$ $z \to y$

So:

① : $v_x = 0$ $\quad a_x = 0$

$v_y = A\omega$ $\quad a_y = 0$

$v_z = A\omega\cos\omega t$ $\quad a_z = -A\omega^2\sin\omega t$

② $a'_x = 0$ (since $a_x = 0$)

$a'_y = 0$ (since $a_y = 0$)

$$a'_z = \frac{a_z}{\gamma^2\left(1 - \frac{u v_x}{c^2}\right)} = \quad // v_x = 0 , \ \gamma = \frac{1}{\sqrt{1 - u^2/c^2}} //$$

$$= - A\omega^2 \left(1 - \frac{u^2}{c^2}\right) \sin \omega t$$

③ transform back:

$a'_x = 0$

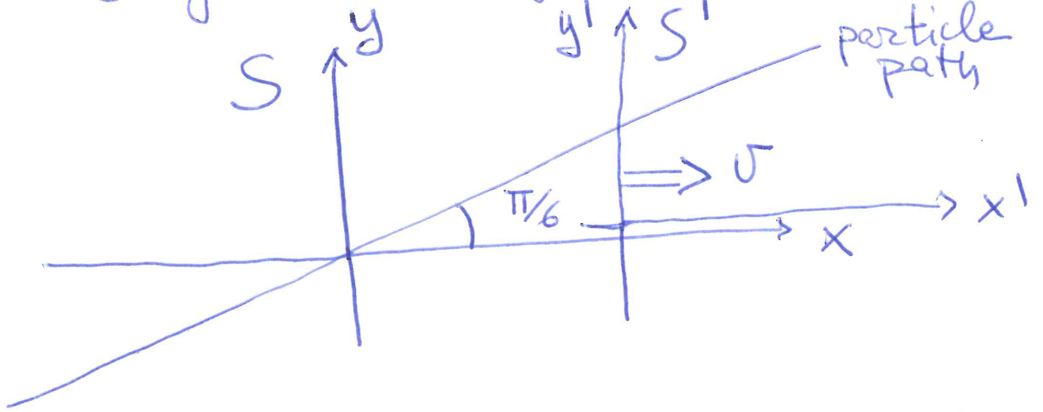
$a'_y = -A\omega^2\left(1 - \frac{u^2}{c^2}\right) \sin\omega t$

$a'_z = 0$.

//

4.2

Assume for simplicity that the particle goes through the origin of S



in S: $v_x = u \times \cos\frac{\pi}{6} = \frac{c}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\cdot c$

$$\Rightarrow \quad x = v_x t = \frac{\sqrt{3}}{4}\cdot ct$$

transforming to S':

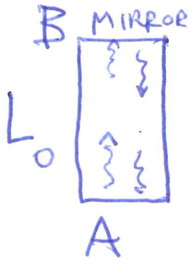
$$x' = \gamma(x - vt) = \gamma\left(\frac{\sqrt{3}}{4}c - v\right)t$$

But if we want the trajectory to be ⊥ to the x' axis, it must be $x' = 0$ all the time.

$$\Rightarrow \quad v = \frac{\sqrt{3}}{4}\cdot c$$

4.3

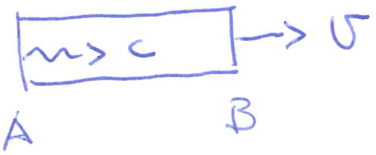
Suppose the clock at rest has length L_0:



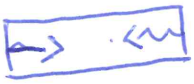
Time between a full trip A→B→A

$$T_0 = \frac{2L_0}{c}$$

ⓐ $L = L_0/\gamma$



$$\text{time } A \to B = \frac{L}{c-v}$$

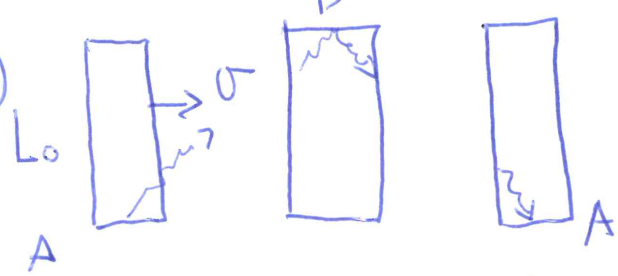


$$\text{time } B \to A = \frac{L}{c+v}$$

$$\text{Total time } T = L\left(\frac{1}{c-v} + \frac{1}{c+v}\right) =$$

$$= \frac{L_0}{\gamma} \cdot \frac{2c}{c^2 - v^2} = \frac{2L_0}{c} \cdot \gamma = T_0 \gamma$$

ⓑ

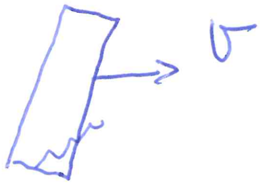


$$\text{Time } A \to B \to A = 2 \times \text{time } A \to B =$$

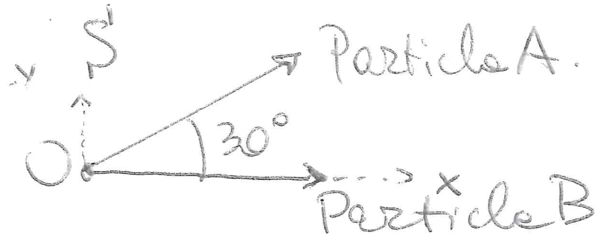
$$= \frac{2L_0}{\sqrt{c^2 - v^2}} = \frac{2L_0}{c} \frac{1}{\sqrt{1 - v^2/c^2}} = T_0 \gamma$$

They are both the same and are ~~tho~~ those predicted.

(c) The same MOST work for any inclination.



4.4



The velocity of particle A in S, chosing O as the origin is $v_x = \frac{\sqrt{3}}{2} v$, $v_y = \frac{1}{2} v$

To get the relative velocity we boost to the rest frame of Particle B.

$$v_x' = \frac{v_x - v}{1 - v \cdot v_x} \qquad v_y' = \frac{v_y}{\gamma(v)(1 - v v_x)} \Rightarrow$$

$$v_x' = \frac{\left(\frac{\sqrt{3}}{2} - 1\right) v}{1 - \frac{\sqrt{3}}{2} v^2} \qquad v_y' = \frac{\frac{1}{2} v}{\gamma(v)\left(1 - \frac{\sqrt{3}}{2} v^2\right)}$$

$$v' = \sqrt{v_x'^2 + v_y'^2} = \frac{1}{1 - \frac{\sqrt{3}}{2} v^2} \cdot \sqrt{\left(\frac{\sqrt{3}}{2} - 1\right)^2 v^2 + \frac{1}{4} v^2 (1 - v^2)}$$

$$= \frac{\sqrt{(8 - 4\sqrt{3}) v^2 - v^4}}{2 - \sqrt{3} v^2} = v \cdot \frac{\sqrt{(8 - 4\sqrt{3}) - v^2}}{2 - \sqrt{3} v^2}$$

4.5

a) The first case deals with two part. not connected with each other.

We do NOT need to do any relativistic "addition" since we are always in the same frame:

$$t = \frac{1m}{0{,}7c - 0{,}6c} = 3{,}3 \times 10^{-8} s$$

b) Here we need to take the length contraction into account:

$$t = \frac{1m/\gamma}{0{,}7c - 0{,}6c} = \frac{0{,}8m}{0.1c} = 2{,}64 \times 10^{-8} s$$

c) They are not the same because in a) the length "1m" refers to our frame. In b) it's the length of the moving rod.

4.6

$x = \frac{k}{3} t^3 \Rightarrow v = k t^2 \Rightarrow a = 2kt$

This can only be valid for

$|v| < c \Rightarrow |t| < \sqrt{c/|k|}$

In this time interval:

$$\alpha = \gamma^3 a = \frac{2kt}{\left(1 - \left(\frac{kt^2}{c}\right)^2\right)^{3/2}}$$

Note that also α does not make sense for $|t| > \sqrt{c/|k|}$.

4.7

$L = 1m$ at rest

The relative velocity between the two ships is

$$V = \frac{2v}{1 + v^2/c^2} \quad \left(\text{addition formula with } u'_x = v\right)$$

So the lenght on the other ship is $L' = L/\gamma(V) = \sqrt{1 - V^2/c^2}\, L$

$L = 1\,m$, $L' = 60\,cm \Rightarrow V = 0.8 \times c$

$\Rightarrow v = 0{,}5\,c$

$\Rightarrow$ Lenght from Earth:

$$L_{Earth} = \sqrt{1 - v^2/c^2}\, L = 86{,}6\,cm.$$

4.8

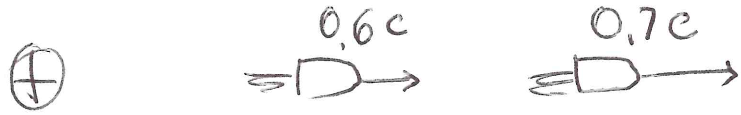
From Earth

We know $$\frac{v+v}{1+v^2/c^2} = 0{,}9\,c$$

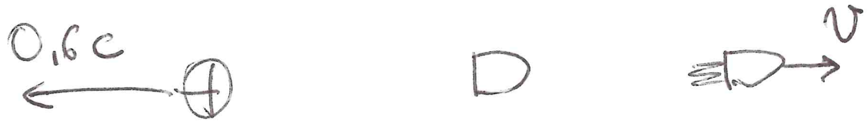
$$\Rightarrow v = 0{,}63\,c$$

4.9

Earth Frame



Boost to



$$v = \frac{0{,}7c - 0.6c}{1 - 0.7 \cdot 0.6 \frac{\cancel{c}^2}{\cancel{c}^2}} = 0{,}17c$$

4.10

$1\,yr \simeq 3.2 \times 10^7 s$

$1\,ly = c \times 1\,yr = 3 \times 10^8 \frac{m}{s} \times 3.2 \cdot 10^7 s =$

$= 9.6 \times 10^{15} m.$

a): $g = 9.8 \frac{m}{s^2} = 9.8 \times \dfrac{\frac{1}{9.6} \times 10^{-15}\, ly}{\left(\frac{1}{3.2}\right)^2 \times 10^{-14}\, yr^2} =$

$= 9.8 \dfrac{(3.2)^2}{9.6} \times 10^{-1}\, ly/yr^2 \simeq 1.\ ly/yr^2.$

b). Set $c=1$ and use units ly, yr. Set also $\alpha = 1$ from a), so we can write in "dimensionless" units:

$$x = \frac{c^2}{\alpha}\left(\sqrt{1+\frac{\alpha^2 t^2}{c^2}} - 1\right) \Rightarrow x = \sqrt{1+t^2} - 1$$

$$v = \frac{\alpha t}{\sqrt{1+\frac{\alpha^2 t^2}{c^2}}} \Rightarrow v = \frac{t}{\sqrt{1+t^2}}$$

We know that $\gamma = \dfrac{1}{\sqrt{1-v^2}} = \dfrac{1}{\sqrt{1-\left(\frac{t}{\sqrt{1+t^2}}\right)^2}}$

$$= \frac{1}{\sqrt{\frac{1}{1+t^2}}} = \sqrt{1+t^2}$$

But we want to know γ as a funct. of τ

So must compute $t(\tau) = ?$

To do that, consider.

$$d\tau^2 = dt^2 - dx^2$$

$$\Rightarrow \quad 1 = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 =$$

$$= \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 \cdot \left(\frac{dx}{dt}\right)^2 =$$

$$= \left(\frac{dt}{d\tau}\right)^2 (1 - v^2) =$$

$$= \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{t^2}{1+t^2}\right) =$$

$$= \left(\frac{dt}{d\tau}\right)^2 \frac{1}{1+t^2}$$

$$\Rightarrow \begin{cases} \frac{dt}{d\tau} = \sqrt{1+t^2} \\ t(0) = 0 \end{cases} \Rightarrow t = sh(\tau)$$

Put back into γ:

$$\gamma = \sqrt{1+t^2} = \sqrt{1+sh^2\tau} = ch\tau.$$

Also note: $x = \sqrt{1+t^2} - 1 = ch\tau - 1.$

	$\tau = 1$ day	1 yr	10 yr
$\gamma =$	1,000004	1,5	≈ 11000
$x =$	0,000004 ly	0,5 ly	≈ 11000 ly
$t =$	0,0027 yr	1,18 yr	≈ 11000 yr

c) The whole trip takes 4×10yr for the crew, but 4×11000 yr for us
the star is about 11000 ly away.

5.1

The full relativistic formula is:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Expand to II order in v:

$$u = u' + \left(1 - \frac{u'^2}{c^2}\right)v - \frac{u'}{c^2}\left(1 - \frac{u'^2}{c^2}\right)v^2$$

$$= u' + kv - u'k\frac{v^2}{c^2}$$

The relative correction is

$$\frac{u'k\, v^2/c^2}{u' + kv} = \frac{\frac{c}{m}\left(1 - \frac{1}{m^2}\right) \cdot \frac{v^2}{c^2}}{\frac{c}{m} + \left(1 - \frac{1}{m^2}\right)v} =$$

$$= \frac{\frac{1}{m}\left(1 - \frac{1}{m^2}\right)\frac{v^2}{c^2}}{\frac{1}{m} + \left(1 - \frac{1}{m^2}\right)\frac{v}{c}} \underset{(v \ll c)}{\simeq} \left(1 - \frac{1}{m^2}\right)\frac{v^2}{c^2} \simeq 5 \cdot 10^{-16}\,\%$$

5.2 In this (very unrealistic!)

(a) case, we must use the exact formula:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\frac{c}{n} + \frac{c}{2}}{1 + \frac{c/n \cdot c/2}{c^2}} =$$

$$= \frac{\frac{1}{n} + \frac{1}{2}}{1 + \frac{1}{2n}} c = \frac{\frac{3}{4} + \frac{1}{2}}{1 + \frac{3}{8}} c = \frac{10}{11} c$$

(b) Fizeau's formula:

$$u = u' + \left(1 - \frac{1}{n^2}\right) v =$$

$$= \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)\frac{c}{2} = \left(\frac{3}{4} + \frac{1}{2}\left(1 - \frac{9}{16}\right)\right) c$$

$$= \frac{31}{32} \cdot c$$

5.3

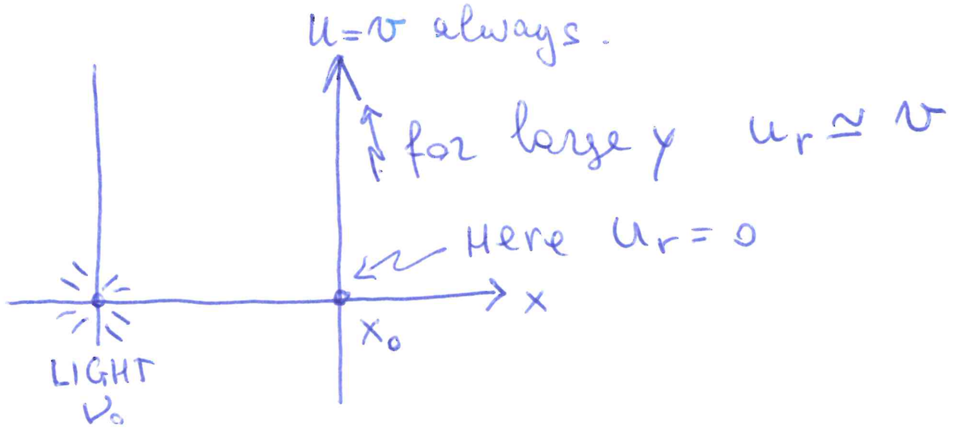
Putting in the values in the problem:

$$v = v_0 \frac{1 - 1/3}{\sqrt{1-(1/2)^2}} = 0,77\, v_0 .$$

5.4

We use the same formula we found in problem 5.3:

$$\nu = \gamma(u)(1 - u_r)\,\nu_0$$



So, near the x axis:

$$\nu = \gamma(v)\,\nu_0 = \frac{\nu_0}{\sqrt{1-v^2}}$$

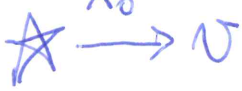
(c = 1 units).

Far from the x axis:

$$\nu = \gamma(v)(1 - v)\,\nu_0 = \sqrt{\frac{1-v}{1+v}}\,\nu_0 .$$

5.5

$$\nu = \frac{c}{\lambda} \qquad\qquad \nu_0 = \frac{c}{\lambda_0}$$



ⓐ Doppler formula :

$$\frac{c}{\lambda_0} = \gamma(v)\left(1+\frac{v}{c}\right)\frac{c}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}}\,\frac{c}{\lambda}$$

$$\Rightarrow \sqrt{\frac{1+v/c}{1-v/c}} = \frac{\lambda}{\lambda_0} \Rightarrow v = \frac{\left(\frac{\lambda}{\lambda_0}\right)^2 - 1}{\left(\frac{\lambda}{\lambda_0}\right)^2 + 1}\,c = 0{,}969\,c$$

very close to c!

ⓑ $$d \simeq \frac{c}{H} = \frac{3\times 10^8\ \mathrm{m/s}}{72\times 10^3\ \frac{\mathrm{m}}{\mathrm{s}\cdot\mathrm{Mpc}}} = 4{,}2\times 10^3\ \mathrm{Mpc}$$

(Note : the size of the "observable universe is $\sim 15\times 10^3$ Mpc).

5.6 (c=1 units)

Exact formula $\frac{\nu_0}{\nu} = \sqrt{\frac{1+u}{1-u}} \overset{\sim}{=} (\text{Taylor})\ 1 + u + \frac{1}{2}u^2$

Non relat. formula: $\frac{\nu_0}{\nu} = 1 + u$.

$$\text{Error} = \frac{\frac{1}{2}u^2}{1+u} \simeq \frac{1}{2}u^2 = \frac{1}{2} \cdot 0{,}01^2 = 0{,}005\%_0$$

5.7

ν

$\nu_0 = 3\nu$

v

$$\frac{\nu_0}{\nu} = 3 = \sqrt{\frac{1+v/c}{1-v/c}}$$

$$\Rightarrow v = 0.8\,c$$

$$\left\langle \frac{1}{2} m v^2 \right\rangle = \frac{3}{2} k_B T$$

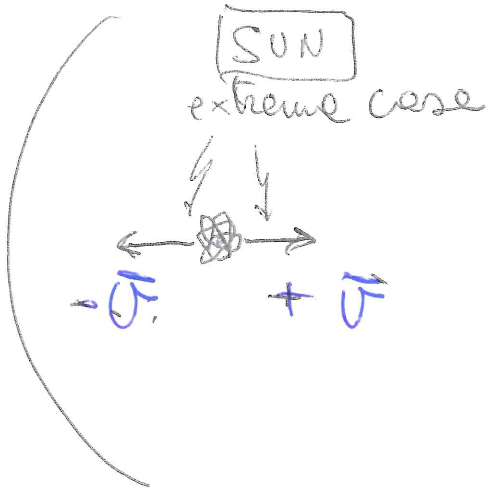
5.8

$$\sqrt{\langle v^2 \rangle} \simeq \sqrt{\frac{3 k_B T}{m}} \simeq 1.2 \times 10^4 \, \frac{m}{s} = \bar{v}$$

Earth $\oplus$

$$\left(\frac{\bar{v}}{c} \sim 4 \times 10^{-5} \right) \sim \sqrt{\frac{3 k_B T}{m c^2}}$$

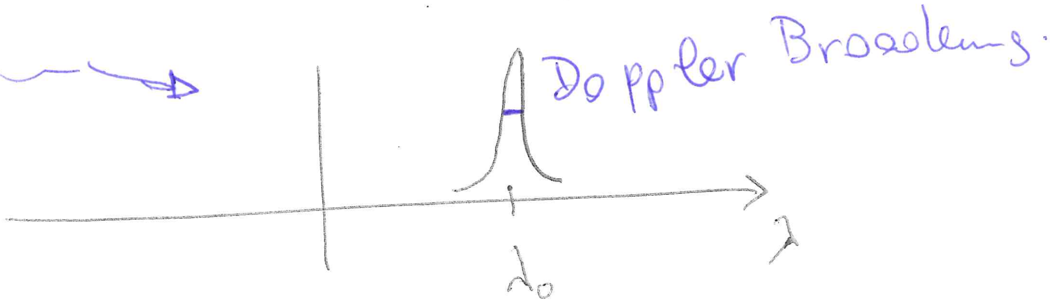
(you can look up the whole distribution in e.g. Wikipedia).



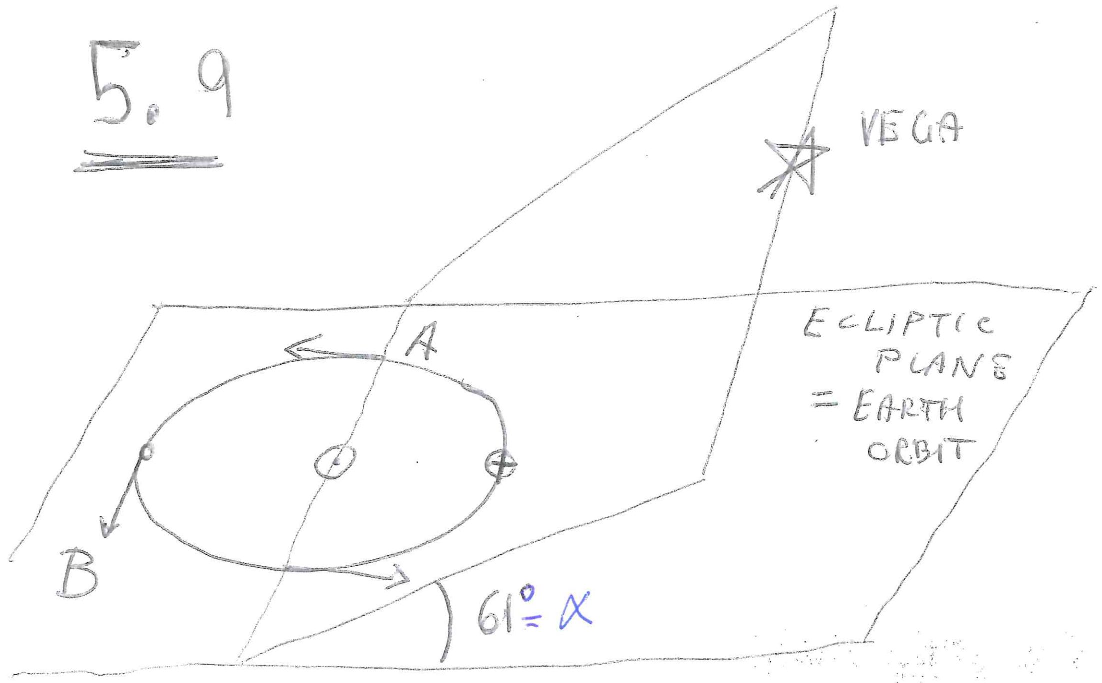
$$\frac{\lambda}{\lambda_0} = \frac{\nu_0}{\nu} = \sqrt{\frac{1 \pm \bar{v}/c}{1 \mp \bar{v}/c}} \cong 1 \pm \frac{\bar{v}}{c}$$

$$\lambda \simeq \lambda_0 \left(1 \pm 4 \times 10^{-5} \right)$$

$\therefore$ There is a broadening due to thermal effects:



5.9



Earth velocity around Sun:

$$v \simeq \frac{2\pi \times 150 \times 10^9 \, m}{1 \, yr} \simeq 3 \times 10^4 \, m/s \ll c$$

Aberration: $\cos\alpha' = \dfrac{\cos\alpha + \frac{v}{c}}{1 + \frac{v}{c}\cos\alpha}$

Set $\alpha' = \alpha + \Delta\alpha \qquad \Delta\alpha \ll 1.$

$$\cos\alpha' = \cos(\alpha + \Delta\alpha) = \cos\alpha \cos\Delta\alpha - \sin\alpha \sin\Delta\alpha$$

$$= // \text{Taylor to } \mathcal{O}(\Delta\alpha) // \ \cos\alpha - \Delta\alpha \sin\alpha$$

Also expand in v/c:

$$\frac{\cos\alpha + v/c}{1 + \frac{v}{c}\cos\alpha} \simeq \left(\cos\alpha + \frac{v}{c}\right)\left(1 - \frac{v}{c}\cos\alpha\right) \simeq$$

$$\simeq \cos\alpha + \frac{v}{c}(1-\cos^2\alpha) + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

$$= \cos\alpha + \frac{v}{c}\sin^2\alpha$$

Comparing:

$$\cancel{\cos\alpha} - \Delta\alpha \sin\alpha \simeq \cancel{\cos\alpha} + \frac{v}{c}\sin^2\alpha$$

$$|\Delta\alpha| = + \frac{v}{c}\sin\alpha .$$

Just take the abs. value and don't worry about the sign.

At point A: $\alpha = 61^\circ$

$$\Rightarrow |\Delta\alpha| = + 10^{-4} \times \sin 61^\circ = + 8{,}7 \times 10^{-5}$$

(to convert to arcseconds, multiply by $\frac{180}{\pi} \times 60 \times 60$:

$$|\Delta\alpha| \simeq + 18''$$

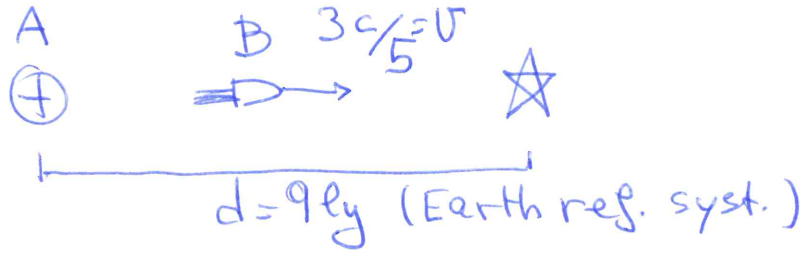
Similarly, in B $\alpha = 90^\circ$

$$\Rightarrow |\Delta\alpha| = + 10^{-4} \simeq 20.6''$$

Parallaxis $\sim \frac{1UA}{d} \sim \frac{8 \text{ min}/c}{25 \text{ ly}/c} = 6{,}1 \times 10^{-7}$ much smaller.

$= 0{,}12''$

5.10



- We must use the Doppler formula because we are dealing with what an observer <u>sees</u>, so we must account for the extra time delay.

Inverting the formula for $\nu \sim \frac{1}{\Delta t}$:

$$\Delta t_A = \sqrt{\frac{1-v/c}{1+v/c}}\,\Delta t_B = \frac{1}{2}\Delta t_B \text{ from A's frame.}$$

So an observer A on Earth sees <u>B</u> move, age, .. half as fast.

THE SAME THING happens for B. She sees A move, age,.. half as fast on her way to the planet.

- B arrives on the planet after $d/v = 9 \cdot \frac{5}{3} = 15$ Earth years <u>but</u> A on Earth sees the arrival after $15 + 9 = 24$ yr (light must come back from the planet.)

- Under the whole 24 yr perod B has aged only $\frac{1}{2} \times 24 = 12$ yr.

- On the way back $v \to -v$:

$$\Delta t_A = \sqrt{\frac{1+v/c}{1-v/c}}\,\Delta t_B = 2\Delta t_B$$

Earth(A) sees B age twice as fast on the way back. THE SAME IS TRUE for B.

- The trip back looks on the monitor only $30-24 = 6$ yr long. (Alternatively $15-9 = 6$yr).
- B has aged $2 \times 6 = 12$ yr on the way back. All together, B has aged $12+12 = 24$ yr against $15+15 = 30$ yr of A.
- From B's point of view the planet is only $d' = d/\gamma = 9 \cdot \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{36}{5}$ ly
- The trip to the planet takes for B: $\frac{36}{5} \times \frac{5}{3} = 12$ yr (as before!)
- During the trip to the planet, B sees A age, move, .. half as fast. When B arrives, A looks $12 \times \frac{1}{2} = 6$yr older

- the trip back takes B exactly the same amount : 12 yr
- During the trip back, B sees A age twice as fast : $12 \times 2 = 24$ yr

So all together A has aged

$6 + 24 = 30$ yr As before!

6.1

a) right. μ is summed (1 up 1 down)
ν in in both left & right side.

b) wrong. There is a "ρ" alone to the right hand side

c) wrong. If I change basis the Tensor changes components

d) right. The exception to c) is if all component are zero. Then they are zero in all frames.

e) right. It says δ is an invariant tensor.

f) wrong. Many indices do not sit right.

6.3

a) In $\epsilon^{\alpha\beta\gamma\delta}$, the indices $\alpha, \beta, \gamma, \delta$ can take values in $\{0,1,2,3\}$.

If there is one repeated index (ex. index 0) then the component vanishes: $\epsilon^{0012} = -\epsilon^{0012}$

$\Rightarrow \epsilon^{0012} = 0$.

If all indices are different, I can use asymmetry to relate them to $\epsilon^{0123} = 1$. Eg. $\epsilon^{2103} = -\epsilon^{2013} =$

$= +\epsilon^{0213} = -\epsilon^{0123} = -1$ and so on..

b) $\epsilon'^{\alpha\beta\gamma\delta} = \Lambda^{\alpha}_{\ \mu} \Lambda^{\beta}_{\ \nu} \Lambda^{\gamma}_{\ \rho} \Lambda^{\delta}_{\ \sigma} \epsilon^{\mu\nu\rho\sigma}$

$= \cancel{\det \Lambda}\, \epsilon^{\alpha\beta\gamma\delta}$

$= \epsilon^{\alpha\beta\gamma\delta}$.

6.4

There are of course many possible solutions. Some examples:

a) $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

e) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0$

6.6

$c=1$ units.

(a) We know: $U^\mu U_\mu = 1$ [1]

$A^\mu U_\mu = 0$ [2]

$A^\mu A_\mu = -\alpha^2$ [3]

In addition, we are given: $\frac{dA^\mu}{d\tau} = \phi U^\mu$.

$\Rightarrow U^\mu = \frac{1}{\phi}\frac{dA^\mu}{d\tau}$. Inserting in [1]:

$$\frac{1}{\phi}\frac{dA^\mu}{d\tau}U_\mu = 0 \Rightarrow \frac{dA^\mu}{d\tau}U_\mu = \phi.$$

taking $\frac{d}{d\tau}$ [2] $\Rightarrow \frac{dA^\mu}{d\tau}U_\mu + A^\mu\frac{dU_\mu}{d\tau} = 0$

$$\Rightarrow \phi + A^\mu A_\mu = 0 \Rightarrow \phi - \alpha^2 = 0$$

$$\Rightarrow \alpha = \sqrt{\phi}.$$

6.7

For constant v ($\Rightarrow$ constant γ):

$$\alpha^2 = \gamma^4 a^2 = \gamma^4 \cdot \left(\frac{v^2}{r}\right)^2$$

For such large γ we can set $v = c$ in the second term.

$$\alpha = \gamma^2 \frac{c^2}{r} = (2 \times 10^5)^2 \times \frac{\left(3 \times 10^8 \frac{m}{s}\right)^2}{4.3 \times 10^3 m} =$$

$$= 8.4 \times 10^{23} m/s^2$$

6.8

$$\begin{cases} x = c_x t \\ y = c_y \sqrt{t} \\ z = 0 \end{cases} \Rightarrow \begin{cases} u_x = c_x \\ u_y = c_y \frac{1}{2\sqrt{t}} \\ u_z = 0 \end{cases} \Rightarrow \begin{cases} a_x = 0 \\ a_y = -\frac{1}{4} c_y t^{-\frac{3}{2}} \\ a_z = 0 \end{cases}$$

$c = 1$ UNITS.

$$\gamma = \frac{1}{\sqrt{1 - c_x^2 - c_y^2/4t}}$$

$$U^\mu = \frac{1}{\sqrt{1-c_x^2 - c_y^2/4t}} \left(1, c_x, c_y \frac{1}{2\sqrt{t}}, 0\right)$$

$$\dot{\gamma} = \frac{-c_y^2/8t^2}{\left(1 - c_x^2 - c_y^2/4t\right)^{\frac{3}{2}}}$$

$$A^\mu = \frac{1}{\sqrt{1-c_x^2-c_y^2/4t}} \left(\frac{-c_y^2/8t^2}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^{\frac{3}{2}}}, \frac{-c_x c_y^2/8t^2}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^{\frac{3}{2}}}, \right.$$

$$\left. \frac{-\frac{c_y^3}{16} t^{-\frac{5}{2}}}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^{3/2}} + \frac{-\frac{1}{4} c_y t^{-\frac{3}{2}}}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^{\frac{1}{2}}}, 0 \right) =$$

$$= \frac{1}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^2} \left(-\frac{c_y^2}{8t^2}, -\frac{c_x c_y^2}{8t^2}, -\frac{1}{4} c_y \left(1-c_x^2\right) \frac{1}{t^{3/2}}, 0 \right)$$

$$U^\mu U_\mu = \frac{1}{1-c_x^2-\frac{c_y^2}{4t}}\left(1-c_x^2-\frac{c_y^2}{4t}\right)=1.$$

$$A^\mu U_\mu = \frac{1}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^{\frac{5}{2}}}\left(-\frac{c_y^2}{8t^2}+\frac{c_x^2c_y^2}{8t^2}+\right.$$

$$\left.+\frac{1}{8}c_y^2(1-c_x^2)\cdot\frac{1}{t^2}\right)=0.$$

7.1

For each proton:

$$T = mc^2\gamma - mc^2 = mc^2\left(\frac{1}{\sqrt{1-0{,}6^2}} - 1\right)$$

$$= 0{,}25\,mc^2$$

Total energy $= 2.5 \times 10^8\, mc^2 =$

$= 2.5 \times 10^8 \times 938\,\text{MeV} =$

$= 2{,}3 \times 10^{11}\,\text{MeV} = 3.7 \times 10^{-2}\,\text{J}.$

7.2

$$T = (\gamma - 1) mc^2 = 10 mc^2$$

$$\Rightarrow \gamma = 11 \Rightarrow v = \sqrt{\frac{120}{121}}\, c \simeq 0{,}996 c$$

$$\left(\gamma = \frac{1}{\sqrt{1-v^2}}, \quad \gamma^2 = \frac{1}{1-v^2}, \quad 1 - v^2 = \frac{1}{\gamma^2} \right.$$

$$\left. v^2 = 1 - \frac{1}{\gamma^2} = \frac{\gamma^2 - 1}{\gamma^2}, \quad v = \sqrt{\frac{\gamma^2 - 1}{\gamma^2}} \right)$$

7.3 Assume the neutrino has mass m_ν and that is the whole reason it arrives 2 hr late (It is not. There are also stellar astrophysical processes involved.).

For the neutrino: $\gamma = \dfrac{E}{m_\nu c^2}$

$$\Rightarrow v = \sqrt{1 - \frac{m_\nu^2 c^4}{E^2}}\, c \simeq //\text{Taylor}// \left(1 - \frac{m_\nu^2 c^4}{2E^2}\right) c$$

The time difference between the arrival of the ν and the photon is:

$$\Delta t = \frac{L}{v} - \frac{L}{c} \hat{=} //\text{Taylor}//$$

$$= \frac{L}{c}\left(1 + \frac{m_\nu^2 c^4}{2E^2}\right) - \frac{L}{c} = \frac{L m_\nu^2 c^3}{2E^2}$$

$$\Rightarrow m_\nu = \sqrt{\frac{2E^2 \Delta t}{L c^3}} = \frac{E}{c^2}\sqrt{\frac{2\Delta t c}{L}}$$

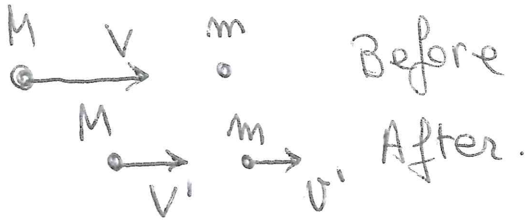
$$= 20\,\text{MeV}/c^2 \cdot \sqrt{\frac{2 \times 2 \cdot \cancel{60} \cdot \cancel{60}\,s \times \cancel{c}}{1.68 \times 10^5 \times 365 \times 24 \times \cancel{60} \times \cancel{60}\,s \times \cancel{c}}}$$

$$= 20\,\mathrm{MeV}/c^2 \times 5.2\times10^{-5} = 1.04\,\mathrm{keV}/c^2 .$$

Note that this is NOT the strongest bound on the ν masse we have!

7.5

[NON REL CASE:



$$\begin{cases} MV = MV' + mv' \\ \frac{1}{2}MV^2 = \frac{1}{2}MV'^2 + \frac{1}{2}mv'^2 \end{cases}$$

Can be solved for $V' = \frac{M-m}{M+m}V$]

One could try the same thing in relativity

Cons. of momentum:

$$MV\gamma(V) = MV'\gamma(V') + mv'\gamma(v')$$

Cons. of energy:

$$M\gamma(V) + m = M\gamma(V') + m\gamma(v')$$

This "could" be solved for V'

(Hint, if you want to do it, set

$$V' = \sqrt{\gamma(V')^2 - 1}/\gamma(V') \qquad v' = \sqrt{\gamma(v')^2 - 1}/\gamma(v')$$

and solve for the γ's first).

BUT the result is a MESS.

TRICK:

Lab

Before: $P^\mu_{lab} = \left(E_{lab} = \sqrt{p^2_{lab}+M^2},\ p_{lab}\right)$ $\quad Q^\mu_{lab} = (m, 0)$.

$M \bullet\!\!\xrightarrow{\ P_{lab}\ }$ $\qquad \overset{\bullet}{m}$

After: $P'^\mu_{lab} = \left(\sqrt{p'^2_{lab}+M^2},\ p'_{lab}\right)$ $\quad Q'^\mu_{lab} = \left(\sqrt{m^2+q'_{lab}},\ q'_{lab}\right)$

$M \bullet\!\!\longrightarrow p'_{lab}$ $\qquad m \bullet\!\!\longrightarrow q'_{lab}$

CM

Before: $P^\mu_{CM} = \left(\sqrt{p^2_{CM}+M^2},\ p_{CM}\right)$ $\quad Q^\mu_{CM} = \left(\sqrt{p^2_{CM}+m^2},\ -P_{CM}\right)$

$M \bullet\!\!\xrightarrow{\ P_{CM}\ }$ $\quad \xleftarrow{\ -P_{CM}\ }\!\!\bullet\ m$

After: $P'^\mu_{CM} = \left(\sqrt{P^2_{CM}+M^2},\ -P_{CM}\right),\ Q'_{CM} = \left(\sqrt{P^2_{CM}+m^2},\ P_{CM}\right)$

$-P_{CM} \longleftarrow\!\!\bullet\ M$ $\qquad m \bullet\!\!\longrightarrow P_{CM}$ (Same P_{CM}. Unique solution).

Consider the LORENTZ INVARIANT quantity $(P'^\mu - Q^\mu)^2$. In the cm frame it is obviously ≥ 0

$$(P'^\mu - Q^\mu)^2 = \left(\sqrt{P^2_{CM}+M^2} - \sqrt{P^2_{CM}+m^2}\right)^2 \cancel{-(P_{CM}-P_{CM})^2} \geq 0$$

In the lab frame: $(P'^\mu - Q^\mu)^2 = P'^{\mu 2} + Q^{\mu 2} - 2P'^\mu Q_\mu$

$$= M^2 + m^2 - 2\sqrt{p'^2_{lab}+M^2}\cdot m = M^2 + m^2 - 2mM\gamma(v')$$

$$\Rightarrow \gamma(v') \leq (M^2+m^2)/2mM.$$

$$\Rightarrow M\gamma(v') \leq (M^2+m^2)/2m$$

7.6

= c=1 units =

ⓐ $\gamma_\pi = \frac{E_\pi}{m_\pi} = \frac{100 \text{ GeV}}{0.140 \text{ GeV}} = 714.$

ⓑ This can be obtained in general as $\gamma_{after} \leq \frac{m_\pi^2 + m_p^2}{2 m_p m_\pi}$.

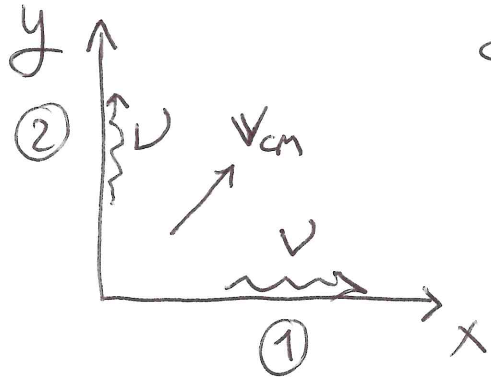
For a proof of this,

(See demonstration in problem 7.5!)

$\Rightarrow \gamma_{after} \leq 3.4 \qquad (\ll 714)$

This means that most of the energy is transmitted to the proton. Relativistic collisions are more "efficient" than the Newtonian ones –

7.7



$c=1$ units.

$$P^{\mu}_{TOT} = P^{\mu}_{1} + P^{\mu}_{2} = (h\nu, h\nu, 0, 0) + (h\nu, 0, h\nu, 0) = (2h\nu, h\nu, h\nu, 0)$$

$$\mathbb{V}_{CM} = \frac{\mathbb{P}_{TOT}}{E_{TOT}} = \frac{(h\nu, h\nu, 0)}{2h\nu} = \frac{1}{2}(1,1,0)$$

Note that $\mathbb{V}_{CM} \cdot \mathbb{V}_{CM} = \frac{1}{4}(1+1) = \frac{1}{2} < 1$.

7.9 Same proceedure as 7.8

Letting $\Lambda \to \pi^+$

c=1

$P \to \mu^+$

$\pi \to \nu$ (man len).

$$P_{(\mu)} = -P_{(\nu)} = \frac{\sqrt{m_{(\mu)}^4 + m_{(\pi)}^4 - 2\, m_{(\mu)}^2 m_{(\pi)}^2}}{2 m_\pi} =$$

$$= \frac{m_{(\pi)}^2 - m_{(\mu)}^2}{2 m_{(\pi)}} = 30{,}6 \text{ MeV}$$

$$E_{(\mu)} = \sqrt{m_{(\mu)}^2 + P_{(\mu)}^2} = 109 \text{ MeV}$$

$$E_{(\nu)} \equiv |P_{(\nu)}| = 30{,}6 \text{ MeV}.$$

7.10 This is a THREE BODY DECAY (NOT a TWO BODY DECAY like the previous exercises).

This means that the final energies are NOT fixed but depend on the relative angles.

The max energy for one of the three particles is attained when the other two move in the opposite direction as a single particle with mass $m_1 + m_2$.

This is a rather intuitive and easy to remember result but it is a bit tricky to prove rigorously (try it!).

Assuming that, the problem is reduced to a 2 body decay:

Case 1: $K^- \to e^- + \underbrace{\pi^0 + \bar{\nu}_e}_{m = m_\pi}$:

$$P_{e^-,max} = \frac{m_K^2 - m^2}{2 m_K} = 231.\ \text{MeV}$$

$$E_{e^-,max} = \sqrt{m_e^2 + P_{e^-max}^2} \simeq P_{e^-max} = 231.\ \text{MeV}$$

Case 2: $\mu^- \to e^- + \underbrace{\nu_\mu + \bar{\nu}_e}_{m = 0}$

$$P_{e^-max} = \frac{1}{2} m_\mu = 52.5\ \text{MeV}$$

$$E_{e^-max} \cong P_{e^-max} = 52.5\ \text{MeV}.$$

7.12

If $\mathbb{P}$ is the 3-mom of one of the photons, its 4-mom must be $p^\mu = (|\mathbb{P}|, \mathbb{P})$ since $p^\mu p_\mu = 0$.

the other photon must have

$p'^\mu = (|\mathbb{P}|, -\mathbb{P})$ since the Higgs

is at rest:

$$(m_{Higgs}, 0) = (|\mathbb{P}|, \mathbb{P}) + (|\mathbb{P}|, -\mathbb{P})$$

$\Rightarrow m_{Higgs} = 2|\mathbb{P}| \simeq$

$2 \times (31.25^2 + 54.13^2 + 0^2)^{\frac{1}{2}} \simeq 125$ GeV.

7, 13

C=1

Case 1: $pp \to ppp\bar{p}$:

$$E_{threshold} = \frac{(4m_p)^2 - m_p^2 - m_p^2}{2m_p} = 7m_p$$

$(\simeq 6.6\,GeV)$

Case 2: $e^+e^- \to p\bar{p}$

$$E_{threshold} = \frac{(2m_p)^2 - m_e^2 - m_e^2}{2m_e} = \frac{2m_p^2 - m_e^2}{m_e}$$

$(\simeq 1.7\ TeV)$

7,14 The energy of a CMBR photon is roughly (they are distributed with a black body distribution).

$$E_{CMBR} \simeq k_B T \simeq 2,3 \times 10^{-4}\ \text{eV}.$$

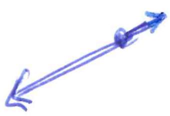
Assume the other photon has energy E_γ.

In a "head on" collision:

$$\underbrace{(E_\gamma, E_\gamma, 0, 0)}_{p^\mu_\gamma} + \underbrace{(E_{CMBR}, -E_{CMBR}, 0, 0)}_{p^\mu_{CMBR}}$$

$$=$$

$e^+ : p^\mu_{e^+}$



e^-

$p^\mu_{e^-}$

So: $P_\gamma^\mu + P_{CMBR}^\mu = P_{e^-}^\mu + P_{e^+}^\mu$

We can compare their squares in any ref. system since they are Lorentz invariant:

$$(E_\gamma + E_{CMBR})^2 - (E_\gamma - E_{CMBR})^2 = (2m_e)^2$$

$$4 E_\gamma E_{CMBR} = 4 m_e^2$$

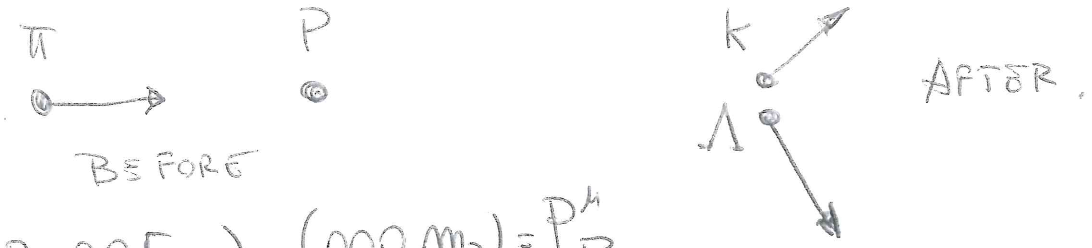
$$\Rightarrow E_\gamma = \frac{m_e^2}{E_{CMBR}} = \frac{(0{,}511)^2 \text{ MeV}^2}{2{,}4 \times 10^{-4} \times 10^{-6} \text{ MeV}}$$

$$\simeq 1{,}1 \times 10^9 \text{ MeV} \quad (!).$$

A similar bound applies to protons (read about the GZK bound).

7.15

$$\boxed{\pi + P \to K + \Lambda}$$



$$\underbrace{P^\mu_\pi = (P_\pi, 0, 0, E_\pi) \quad (0\,0\,0\,m_p) = P^\mu_p}_{\text{in the LAB frame}}$$

The MINIMUM energy of K and Λ in the CM frame is their rest mass

$$\underbrace{P^\mu_K = (0\,0\,0\,m_K) \quad P^\mu_\Lambda = (0\,0\,0\,m_\Lambda)}_{\text{IN the CM frame.}}$$

$$\underbrace{(P^\mu_\pi + P^\mu_p)^2 = (P^\mu_K + P^\mu_\Lambda)^2}_{\text{lorentz invariant, valid in ANY FRAME.}}$$

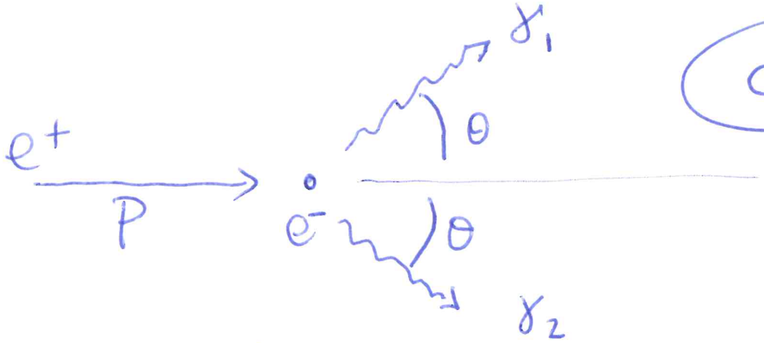
$$(E_\pi + m_p)^2 - P_\pi^2 = (m_K + m_\Lambda)^2$$

$$m_p^2 + E_\pi^2 + 2m_pE_\pi - P_\pi^2 = (m_K + m_\Lambda)^2$$

$$m_p^2 + 2m_pE_\pi + m_\pi^2 = (m_K + m_\Lambda)^2$$

$$E_\pi = \frac{(m_K + m_\Lambda)^2 - m_\pi^2 - m_p^2}{2m_p} = 909 \text{ MeV.}$$

7.16



$c=1$

$P^\mu_{e^+} = (K+m_e, P, 0\ 0)$ $\quad K + m_e = \sqrt{m_e^2+P^2}$

$P^\mu_{e^-} = (m_e, 0, 0, 0)$

$P^\mu_{\gamma_1} = (E, E\cos\theta, E\sin\theta, 0)$

$P^\mu_{\gamma_2} = (E, E\cos\theta, -E\sin\theta, 0)$

$P^\mu_{e^+} + P^\mu_{e^-} = P^\mu_{\gamma_1} + P^\mu_{\gamma_2} \Rightarrow$

$$\begin{cases} K + m_e + m_e = 2E \\ P = 2E\cos\theta \end{cases}$$

$\Rightarrow E = m_e + \frac{K}{2} = 1.011\ \text{MeV}$

from $K + m_e = \sqrt{m_e^2+p^2} \Rightarrow p = \sqrt{K^2+2Km_e}$

$\Rightarrow p = 1.42\ \text{MeV}$

$\Rightarrow \cos\theta = P/2E = 0.70 \Rightarrow \theta \simeq 45°.$

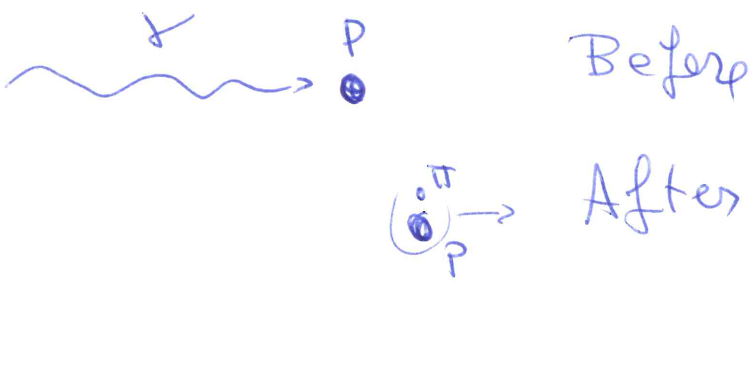
7.17 $\boxed{c=1}$ $\gamma p \to \pi^0 p$

$$E_{threshold} = \frac{(m_{\pi^0} + m_p)^2 - 0 - m_p^2}{2 m_p} =$$

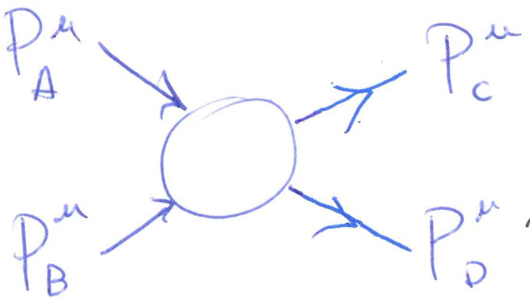
(a)

$$= \frac{m_{\pi^0}^2 + 2 m_p m_{\pi^0}}{2 m_p} \simeq 145 \text{ MeV}$$

(b) Just above threshold the outgoing particles have almost no relative velocity, i.e. they move almost together as a particle of mass $m_{\pi^0} + m_p$ and momentum = incoming photon momentum.



7,18



Reversed sign compared to compendium ($P^\mu_{C,D} \to -P^\mu_{C,D}$)

$$P_A^\mu + P_B^\mu = +P_C^\mu + P_D^\mu$$

$s = (P_A + P_B)^2$ is the Energy^2 in the center of mass AND it is lorentz invariant $\Rightarrow$ $s > 0$ in all frames.

Now:
$$s + t + u = P_A^2 + 2P_A P_B + P_B^2 +$$
$$+ P_A^2 - 2P_A P_C + P_C^2 +$$
$$+ P_B^2 - 2P_B P_C + P_C^2 =$$

$$= 2m_A^2 + 2m_B^2 + 2m_C^2 + 2P_A P_B - 2P_A P_C - 2P_B P_C \quad [1]$$

But also, writing the conservation of momentum as: $P_A^\mu + P_B^\mu - P_C^\mu = P_D^\mu$ and doing the lorentz square:

$$P_A^2 + P_B^2 + P_C^2 + 2P_A \cdot P_B - 2P_A \cdot P_C - 2P_B \cdot P_C = P_D^2$$

$$\Rightarrow 2P_A \cdot P_B - 2P_A \cdot P_C - 2P_B \cdot P_C = m_D^2 - m_A^2 - m_B^2 - m_C^2 \quad [2]$$

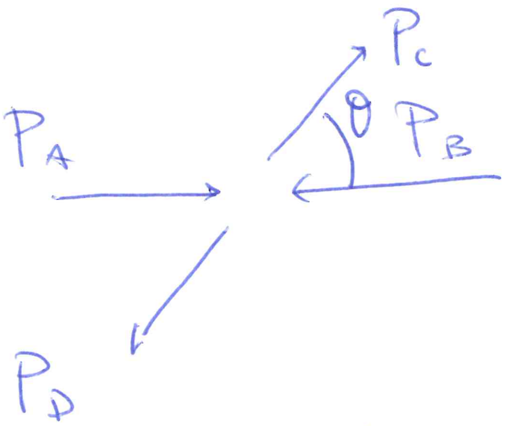
Substituting [2] into [1] we get:

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 .$$

Let now all masses be equal to m and compute t in the CM frame (it is a Lorentz invariant).

Chose $P_A^\mu = (E, P, 0, 0)$ by rotation

$$\Rightarrow P_C^\mu = (E, p\cos\theta, p\sin\theta, 0)$$



$$t = (P_A - P_C)^2 = (0, \underbrace{p(1-\cos\theta), -p\sin\theta, 0}_{\text{SPACE LIKE !}})^2 =$$

$$= -p^2(1-\cos\theta)^2 - (-p\sin\theta)^2 =$$

$$= -2p^2(1-\cos\theta) \leq 0$$

Same for u by exchange $C \leftrightarrow D$.

7.19

We can write $v = 0{,}4\,c$ and set $c = 1$ everywhere.

The rocket equation for a photon exhaust is then:

$$\frac{M_{\text{before}}}{M_{\text{after}}} = \left(\frac{1+v}{1-v}\right)^{\frac{1}{2}} = 1{,}53$$

7.20

Same equation as problem 7.19:

$$\frac{M_{before}}{M_{after}} = \sqrt{\frac{1+v}{1-v}}$$

Now we want $\frac{M_{before}}{M_{after}} = 2$

$$\Rightarrow \quad v = \frac{3}{5} \ (\times c) = 1{,}8 \times 10^{8} \ m/s.$$

7.21

$$E = h\nu = \hbar\omega$$
$$\mathbb{P} = h\mathbb{k} = \hbar\mathbb{K}$$

a) $$E^2 - \mathbb{P}^2 = m^2 = \hbar^2(\omega^2 - \mathbb{K}^2)$$

$$\Rightarrow \omega^2 - \mathbb{K}^2 = \frac{m^2}{\hbar^2} = \frac{4\pi^2}{\ell^2}$$ Put back the "c" by dim. analysis.

b) $$\frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\left(\sqrt{4\pi^2/\ell^2 + k^2}\right) =$$

$$= \frac{1}{2}\frac{2k}{\left(\frac{4\pi^2}{\ell^2} + k^2\right)^{\frac{1}{2}}} = \frac{k}{\omega} = \frac{P}{E} = u.$$

Also as a vector:

$$\nabla_{\mathbb{K}}\,\omega = \frac{\mathbb{K}}{\omega} = \frac{\mathbb{P}}{E} = \mathbb{u}.$$

7.22

$\epsilon = h\nu \simeq 10^{-3}\,eV = 10^{-12}\,GeV$

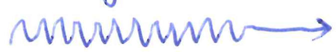
BEFORE :

Proton CMPR photon

$P_p^\mu = (E, P, 0, 0)$ $P_\gamma^\mu = (\epsilon, -\epsilon, 0, 0)$

AFTER :

$q_\gamma^\mu = (\epsilon', \epsilon', 0\ 0)$



q_p^μ (not needed)

$$P_p^\mu + P_\gamma^\mu = q_\gamma^\mu + q_p^\mu$$

$$(P_p^\mu + P_\gamma^\mu - q_\gamma^\mu)^2 = (q_p^\mu)^2$$

$$P_p^2 + P_\gamma^2 + q_\gamma^2 + 2P_p \cdot P_\gamma - 2P_p \cdot q_\gamma - 2P_\gamma \cdot q_\gamma = q_p^2$$

$$m^2 + 0 + 0 + 2(E+P)\epsilon - 2(E-P)\epsilon' - 4\epsilon\epsilon' = m^2$$

// Note: $P_p \cdot P_\gamma = E\epsilon - P\cdot(-\epsilon) = (E+P)\epsilon$

$P_p \cdot q_\gamma = E\epsilon' - P\epsilon' = (E-P)\epsilon'$

$P_\gamma \cdot q_\gamma = \epsilon\epsilon' - (-\epsilon)\epsilon' = 2\epsilon\epsilon'$ //

Solve for ϵ':

$$\epsilon' = \frac{(E+P)\epsilon}{2\epsilon + E - P}$$

Now: $E^2 = p^2 + m^2 \Rightarrow p = \sqrt{E^2 - m^2}$

$= (\text{Taylor})\ E - \frac{m^2}{2E}$ $(m \simeq 1\,\text{GeV})$ proton mass.

$$\Rightarrow \epsilon' \simeq \frac{2E\epsilon}{2\epsilon + \frac{m^2}{2E}} =$$

$$= \frac{2\times 10^{11}\,\text{GeV} \times 10^{-12}\,\text{GeV}}{2\times 10^{-12}\,\text{GeV} + \frac{1\,\text{GeV}^{\cancel{2}}}{2\times 10^{11}\,\cancel{\text{GeV}}}} =$$

$$= \frac{2\times 10^{-1}\,\text{GeV}}{2\times 10^{-12} + 5\times 10^{-12}} \sim 0{,}3\times 10^{11}\,\text{GeV}.$$

$$\sim 3\times 10^{19}\,\text{eV}.$$

8.2

Let us write $\mathbb{F} = q(\mathbb{E} + \mathbb{v} \times \mathbb{B})$.

Recall that $\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d}{dt}(1-v^2)^{-\frac{1}{2}} =$

$$= -\frac{1}{2}(1-v^2)^{-\frac{3}{2}} \times (-2v\dot{v}) = \gamma^3 v\dot{v} = \gamma^3 \mathbb{v}\dot{\mathbb{v}}$$

Also recall that m is a constant.

$$\frac{d}{dt}(\gamma m \mathbb{v}) = \dot{\gamma} m \mathbb{v} + \gamma m \dot{\mathbb{v}} = \mathbb{F}$$

$$= \gamma^3 m(\mathbb{v}\cdot\dot{\mathbb{v}})\mathbb{v} + \gamma m \dot{\mathbb{v}} = \mathbb{F} \quad [1]$$

Take the $\cdot$ product of [1] with $\mathbb{v}$:

$$\gamma^3 m\, \mathbb{v}\cdot\dot{\mathbb{v}}\, v^2 + \gamma m\, \mathbb{v}\cdot\dot{\mathbb{v}} = \mathbb{v}\cdot\mathbb{F}$$

$$\gamma m\, \mathbb{v}\cdot\dot{\mathbb{v}}\,(\underbrace{\gamma^2 v^2 + 1}_{=\gamma^2}) = \mathbb{v}\cdot\mathbb{F} = q\,\mathbb{v}\cdot\mathbb{E}$$

(since $\mathbb{v}\cdot\mathbb{v}\times\mathbb{B} = 0$)

$$\Rightarrow \gamma^3 m\, \mathbb{v}\cdot\dot{\mathbb{v}} = q\,\mathbb{v}\cdot\mathbb{E}$$

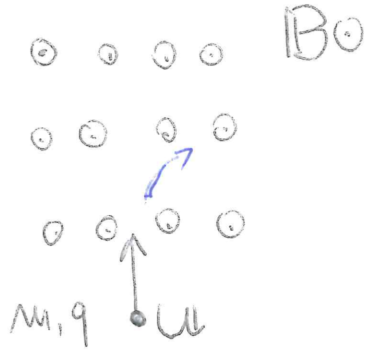
Substituting back in [1]:

$$q(\mathbb{v}\cdot\mathbb{E})\mathbb{v} + \gamma m \dot{\mathbb{v}} = \mathbb{F}$$

$$\Rightarrow \gamma m \dot{\mathbb{v}} = \mathbb{F} - q(\mathbb{v}\cdot\mathbb{E})\mathbb{v} \quad \checkmark$$

$$\gamma \frac{d m\mathbb{v}}{dt} = q(\mathbb{E} + \mathbb{v}\times\mathbb{B} - (\mathbb{v}\cdot\mathbb{E})\mathbb{v}).$$

8.3



$$\mathbf{f} = q\,\mathbf{u} \times \mathbf{B}$$

ⓐ

Relativistic 3-mom. and 3-force.

Once again: $F^{\mu} = \frac{dP^{\mu}}{d\tau} = \gamma\left(\frac{dE}{dt}, \frac{d\mathbf{P}}{dt}\right) = \gamma\left(\frac{dE}{dt}, \mathbf{f}\right)$

$$F^{\mu} = \frac{dP^{\mu}}{d\tau} = m\frac{dU^{\mu}}{d\tau} + \frac{dm}{d\tau}U^{\mu}$$

PRESERVING the rest mass! $\quad \frac{dm}{d\tau} = 0$

$\Rightarrow F^{\mu}U_{\mu} = 0 \Rightarrow \frac{dE}{dt} = \mathbf{u}\cdot\mathbf{f}$ gäller för alla $\mathbb{B}$ bara $\mathbb{E} = 0$.

For this case $\mathbf{u}\cdot\mathbf{f} = 0 \Rightarrow E$ conserved.

also $\mathbf{f} = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}(E\mathbf{u}) = E\cdot\frac{d\mathbf{u}}{dt} = E\,\mathbf{a}$.

thus $\mathbf{a}\cdot\mathbf{u} \propto \mathbf{f}\cdot\mathbf{u} = 0 \Rightarrow u$ constant.

$\left(\frac{du^2}{dt} = 2u\dot{u} = 2\mathbf{u}\cdot\mathbf{a} = 0\right)$

Now the eq. of motion is reduced to the same eq. we have non rel. but with the mass replaced by the (constant) Energy E/c^2)

$$\Rightarrow \quad q\,\mathbf{u}\times\mathbf{B} = \overset{m\gamma}{\overset{\|}{E}}\,\mathbf{a} \quad \Longrightarrow \quad quB = m\gamma\frac{u^2}{r} \Rightarrow qB = m\gamma\cdot\frac{2\pi}{T} \qquad (\mathbf{B} = B\,\mathbf{e}_z)$$

Take the ansatz $\mathbf{u} = \underset{\substack{\Uparrow \\ \text{const.}}}{u}(\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y)$

$$\Rightarrow \mathbf{a} = u\dot{\theta}(\cos\theta\,\mathbf{e}_x - \sin\theta\,\mathbf{e}_y)$$

$$\Rightarrow quB(-\sin\theta\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_x) = Eu\dot{\theta}(\cos\theta\,\mathbf{e}_x - \sin\theta\,\mathbf{e}_y)$$

$$\Rightarrow \dot{\theta} = \frac{qB}{E} = \omega \text{ also constant, } T = \frac{2\pi E}{qB}$$

integrating $\mathbf{u} = u(\sin\omega t\,\mathbf{e}_x + \cos\omega t\,\mathbf{e}_y)$

gives $\mathbf{r} = \frac{u}{\omega}(-\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y)$

$$\Rightarrow \text{Radius } R = \frac{u}{\underset{\frac{2\pi}{T} =}{\omega}} = \frac{uE}{qB} \text{ also const.}$$

Putting units back and writing $m\gamma c^2$ instead of E:

$$R = \frac{\cancel{c}\,m u \gamma(u)}{qB}, \qquad T = \frac{2\pi \cancel{c}\,m\gamma(u)}{qB}$$

Coulomb = Ampere × seconds

[Note that the formula for $\gamma = 1$ can be obtained by non rel. considerations

$$m\frac{v^2}{R} = qB\cancel{v} \Rightarrow \omega = \frac{qB}{m}.$$
]

If the particle has a u_z component such component remains const. since $(u \times B)_z = 0$ still.

$\Rightarrow$ helix w/ same radius.

(b) From $R = \frac{\cancel{c}\, m u \gamma}{qB}$ we get $R = \frac{\cancel{c}P}{qB} \Rightarrow \cancel{c}P = q \cdot RB$

$P = qBR$

Remember that $1\,\text{Tesla} = \frac{1\,\text{kg} \times}{1\,\text{Coulomb} \times s}$

$F = qvB$ in S.I.

So to get the units right I have to multiply the RHS by an extra "c":

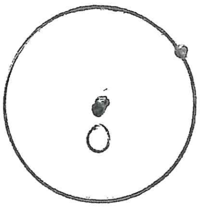
$$cP = qcRB = 1.6 \times 10^{-19} C \times 3 \times 10^{8} \frac{m}{s} \times \frac{R}{\text{meter}} \times m \times \frac{B}{\text{Tesla}} \times T$$

$$= 4.8 \times 10^{-11} C \cdot \frac{m^2 T}{s} \frac{R}{\text{meter}} \times \frac{B}{\text{Tesla}} =$$

$$= 4.8 \times 10^{-11} J \frac{R}{\text{meter}} \cdot \frac{B}{\text{Tesla}} = 300\,\text{MeV} \frac{R}{\text{meter}} \frac{B}{\text{Tesla}},$$

8.4

$c=1$



$\hat{x} = R\cos\omega\hat{t}$

$\hat{y} = R\sin\omega\hat{t}$ $\quad (v = \omega R)$

$\hat{z} = 0$

$$R^\mu = \left(t-\hat{t}, -R\cos\omega\hat{t} + 0, -R\sin\omega\hat{t} + 0, 0\right)$$

$$R^\mu R_\mu = 0 \Rightarrow \hat{t} = t - R \quad (\hat{t} < t)$$

$$U^\mu = \gamma\left(1, -v\sin\omega\hat{t}, v\cos\omega\hat{t}, 0\right)$$

$$R_\mu U^\mu = \gamma\left(+R + Rv\cancel{\sin\omega\hat{t}\cos\omega\hat{t}} - \cancel{Rv\sin\omega\hat{t}\cos\omega}\right)$$

$$A^\mu = \frac{qU^\mu}{4\pi R_\mu U^\mu} \qquad A^0 = V = \frac{q\cancel{\gamma}}{4\pi\cancel{\gamma}R} = \frac{q}{4\pi R} \text{ constant}$$

while, eg. $A^1 = A_x = \dfrac{q\cancel{\gamma}(-v\sin\omega\hat{t})}{4\pi\cancel{\gamma}R} =$

$$= -\frac{qv}{4\pi R}\sin\omega(t-R)$$

8.6 ρ_0 is scalar and U^μ is a 4-vector

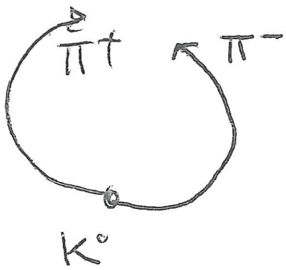
$\Rightarrow \rho_0 U^\mu$ is a 4vector.

8.7 Using the formula in problem 8.3.

$$\frac{cp}{\text{MeV}} = 300 \times \frac{B}{\text{Tesla}} \times \frac{R}{\text{meter}}$$

$$\Rightarrow cp = 300 \times 2 \times 0{,}344 \text{ MeV}$$

$$= 206 \text{ MeV}.$$



$$E_\pi = \sqrt{(cp)^2 + (m_\pi c^2)^2} = 249 \text{ MeV}$$

$$m_{K^\circ} = 2E_\pi = 498 \text{ MeV}$$

8.9 $\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) =$

$= \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = 0$ (a)

(b) set $A_\mu = \epsilon_\mu e^{ik\cdot x}$

$\partial^\mu A_\mu = i\, k^\mu \epsilon_\mu e^{ik\cdot x} = 0$

$\partial^2 A_\nu = (-ik^\mu)(-ik_\mu) A_\nu =$

$= -k^\mu k_\mu A_\nu = 0$

$\Rightarrow \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = 0$

(c) $F_{\mu\nu} = i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu) e^{ik\cdot x}$

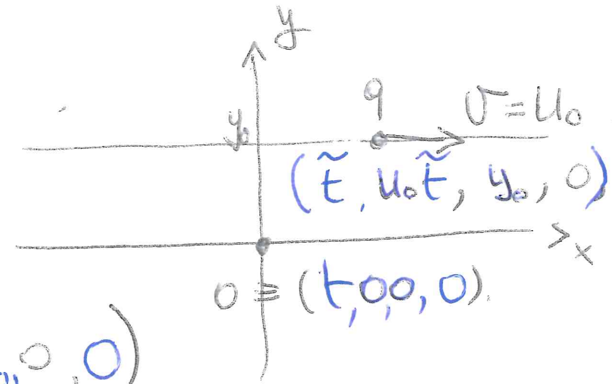
Let $\epsilon_\rho \to \epsilon_\rho + \alpha k_\rho$

$F_{\mu\nu} \to i(k_\mu(\epsilon_\nu + \alpha k_\nu) - k_\nu(\epsilon_\mu + \alpha k_\mu))e^{ik}$

$= i(k_\mu \epsilon_\nu + \cancel{\alpha k_\mu k_\nu} - k_\nu \epsilon_\mu - \cancel{\alpha k_\nu k_\mu})e^{ikx}$

$= F_{\mu\nu}$ ✓

8.10



$$U^\mu = \gamma(u_0)(1, u_0, 0, 0)$$

$$R^\mu = (t-\tilde t, -u_0\tilde t, -y_0, 0)$$

where $\tilde t$ is the time at which the charge affects the origin:

$$R_\mu R^\mu = (t-\tilde t)^2 - y_0^2 - u_0^2\tilde t^2 = 0$$

$$\Rightarrow \tilde t = \frac{t - \sqrt{t^2-(1-u_0^2)(t^2-y_0^2)}}{1-u_0^2}$$

$$\phi^0 = \frac{q}{4\pi}\frac{\gamma}{\gamma(t-\tilde t)+\gamma u_0^2\tilde t} = \frac{q/4\pi}{t-\tilde t+u_0^2} =$$

where $\tilde t$ is given by the eq. above.

$$= \frac{q/4\pi}{\sqrt{t^2-(1-u_0^2)(t^2-y_0^2)}} = \frac{q/4\pi}{\sqrt{u_0^2t^2+(1-u_0^2)y_0^2}}$$

8.11

obvious

μ, ν are dummy indices

a) $$T_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu}F^{\mu\nu} + T_{\mu\nu}F^{\mu\nu}\right) =$$

$$= \frac{1}{2}\left(T_{\mu\nu}F^{\mu\nu} + T_{\nu\mu}F^{\nu\mu}\right) =$$ use symmetry/antisym.

$$= \frac{1}{2}\left(T_{\mu\nu}F^{\mu\nu} + T_{\mu\nu}(-F^{\mu\nu})\right) =$$

$$= \frac{1}{2}\left(T_{\mu\nu}F^{\mu\nu} - T_{\mu\nu}F^{\mu\nu}\right) = 0.$$

Equivalently, you can just say $T_{\mu\nu}F^{\mu\nu} \underset{\text{(dummy)}}{=} T_{\nu\mu}F^{\nu\mu} \underset{\text{(symmetry)}}{=}$

$$= T_{\mu\nu}(-F^{\mu\nu}) = -T_{\mu\nu}F^{\mu\nu} \Rightarrow T_{\mu\nu}F^{\mu\nu} = 0.$$

b) I need to show that $X^{\nu\mu} = F^{\nu\rho}F_{\rho}{}^{\mu}$ is symmetric then I can use a).

$$X^{\mu\nu} = F^{\mu\rho} F_\rho{}^\mu = - F^{\rho\mu} F_\rho{}^\nu =$$

$$= - F_\rho{}^\mu F^{\rho\nu} = - F_\rho{}^\mu (- F^{\nu\rho}) =$$

$$= + F_\rho{}^\mu F^{\nu\rho} = F^{\nu\rho} F_\rho{}^\mu = X^{\nu\mu}$$

Thus: $F_{\mu\nu} X^{\mu\nu} = 0$.

c) I need to show that

$Y^{\nu\mu} = F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}$ is antisymmetric

then I can use a).

$$Y^{\mu\nu} = F^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} = \text{//dummy indices//}$$

$$= F^{\mu\sigma} F_{\sigma\rho} F^{\rho\nu} = F^{\rho\nu} F_{\sigma\rho} F^{\mu\sigma}$$

$$= \text{//antisymm.//} (-F^{\nu\rho})(-F_{\rho\sigma})(-F^{\sigma\mu})$$

$$= - F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} = - Y^{\nu\mu}$$

Thus $T_{\mu\nu} Y^{\mu\nu} = 0$.

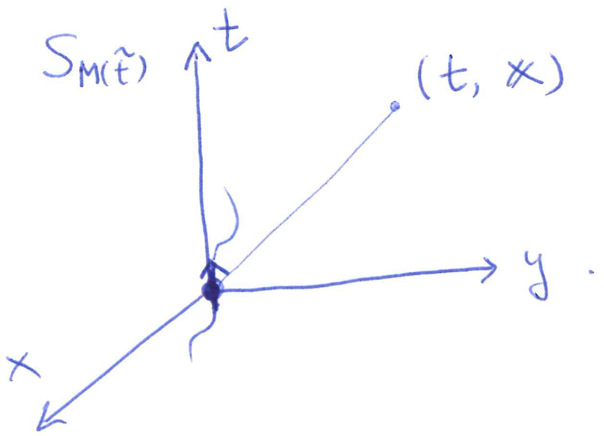
8.12

Go to the rest frame of the charge at time $\tilde{t}$ (the instantaneous rest frame).

In that frame, (still using $t, \mathbb{x}$):

$$V(t, \mathbb{x}) = \frac{q/4\pi}{|\mathbb{r}(\tilde{t}) - \mathbb{x}|} \quad , \quad \mathbb{A}(t, \mathbb{x}) = 0$$

where $t = \tilde{t} + |\mathbb{r}(\tilde{t}) - \mathbb{x}|$.



Define the 4-vectors:

$$R^\mu = (t - \tilde{t}, \mathbb{x} - \mathbb{r}(\tilde{t}))$$

$$U^\mu = \gamma(1, \dot{\mathbb{r}}(\tilde{t})).$$

$$R^\mu R_\mu = 0 \longrightarrow t = \tilde{t} + |\mathbb{r}(\tilde{t}) - \mathbb{x}|$$

$$\frac{1}{4\pi}\frac{qU^\mu}{R^\nu U_\nu} = A^\mu \longrightarrow V = \frac{q/4\pi}{|\mathbb{r}(\tilde{t}) - \mathbb{x}|} \quad , \quad \mathbb{A} = 0.$$