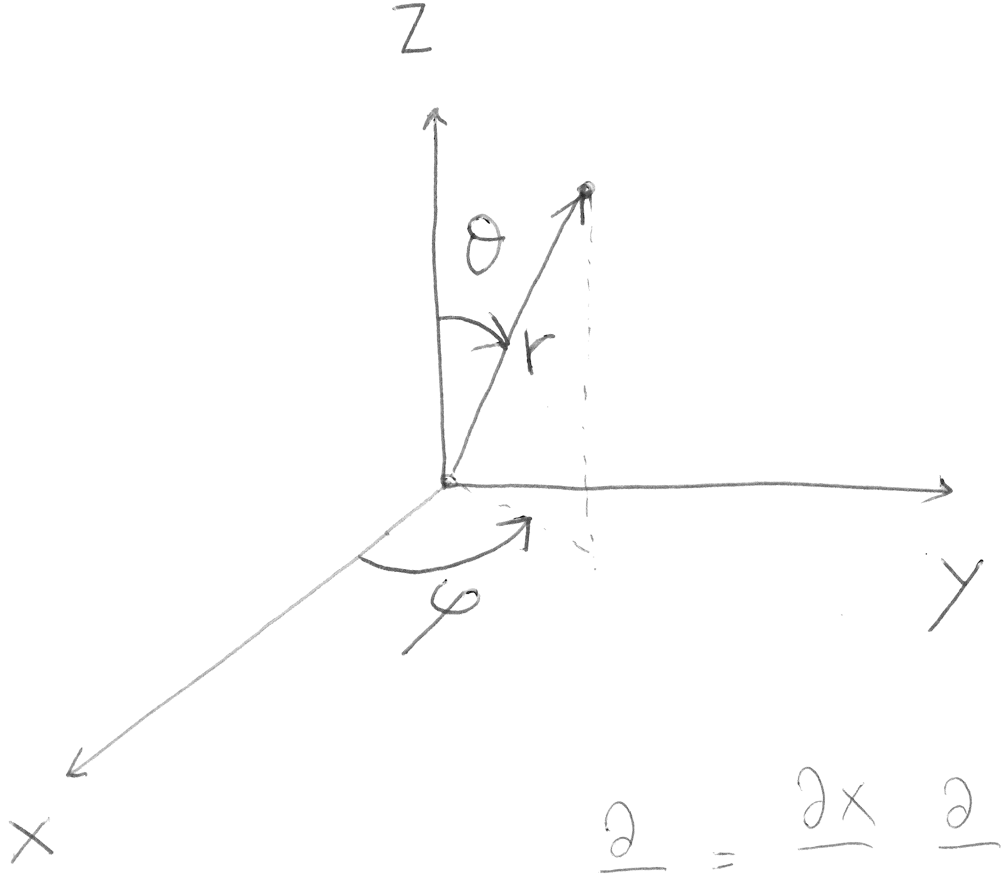
CENTRAL POTENTIAL ($\hbar = 1$!) IN 3 DIMENSIONS



$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} \quad \text{etc...}$$

shorthand : $\frac{\partial}{\partial r} = \partial_r$, etc...

$$\partial_r = \sin\theta\cos\varphi\,\partial_x + \sin\theta\sin\varphi\,\partial_y + \cos\theta\,\partial_z$$

$$\partial_\theta = r\cos\theta\cos\varphi\,\partial_x + r\cos\theta\sin\varphi\,\partial_y - \sin\theta\,\partial_z$$

$$\partial_\varphi = -r\sin\theta\sin\varphi\,\partial_x + r\sin\theta\cos\varphi\,\partial_y + 0$$

in matrix notation:

$$\begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r\sin\theta}\partial_\varphi \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}}_{\text{ORTHOGONAL ! } \Omega^{-1} = \Omega^T} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r\sin\theta}\partial_\varphi \end{pmatrix}$$

$$L_x = yP_z - zP_y = -i\left(y\partial_z - z\partial_y\right) = \dots = i\left(\sin\varphi\,\partial_\theta + \frac{\cos\theta}{\sin\theta}\cos\varphi\,\partial_\varphi\right)$$

$$L_y = zP_x - xP_z = -i\left(z\partial_x - x\partial_z\right) = \dots = i\left(-\cos\varphi\,\partial_\theta + \frac{\cos\theta}{\sin\theta}\sin\varphi\,\partial_\varphi\right)$$

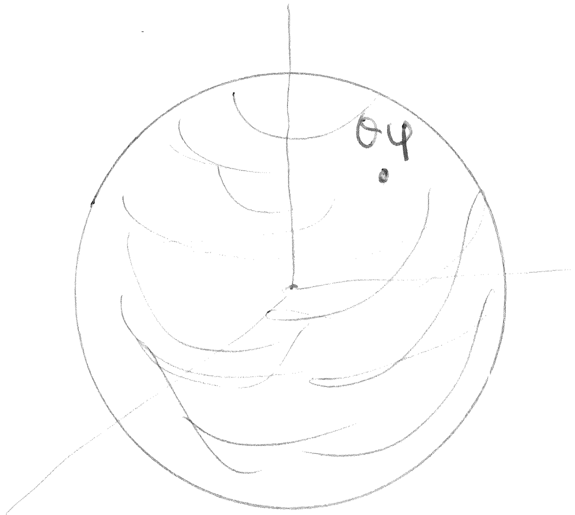
$$L_z = xP_y - yP_x = -i\left(x\partial_y - y\partial_x\right) = \dots = -i\,\partial_\varphi$$

$$\mathbb{L}^2 \equiv L_x^2 + L_y^2 + L_z^2 = -\left(\frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right)\right)$$

$$[L_x, L_y] = i L_z \,, \quad [L_y, L_z] = i L_x \,, \quad [L_z, L_x] = i L_y .$$

$\Rightarrow \exists$ wave functions $Y_{\ell,m}(\theta,\varphi)$ (r = fixed!)

s.t. $\mathbb{L}^2 Y_{\ell,m} = \ell(\ell+1)$ $\qquad L_z Y_{\ell,m} = m Y_{\ell m}$.



particle on the surface of the sphere.

$$\psi(\theta,\varphi) = \sum_{\ell,m} c_{\ell m} Y_{\ell m}(\theta,\varphi) .$$

$$L_z = -i\,\partial_\varphi$$

$$L_z Y_{\ell m}(\theta,\varphi) = m\, Y_{\ell m}(\theta,\varphi)$$

$$-i\,\partial_\varphi Y_{\ell m}(\theta,\varphi) = m\, Y_{\ell m}(\theta,\varphi)$$

$$\Rightarrow Y_{\ell m}(\theta,\varphi) = f_{\ell m}(\theta)\cdot e^{im\varphi} \qquad m \in \mathbb{Z} \Rightarrow \ell = 0, \cancel{\tfrac{1}{2}}, 1, 2 \ldots$$

$$L_i = \begin{pmatrix} \boxed{0} & & & \\ & \boxed{\ell=1} & & 0 \\ & & \boxed{\ell=2} & \\ & 0 & & \ddots \end{pmatrix} \begin{array}{l} Y_{00} \\ \left.\begin{array}{l} Y_{1-1} \\ Y_{10} \\ Y_{11} \end{array}\right\} \\ \left.\begin{array}{l} \vdots \end{array}\right\} \end{array}$$

$Y_{00}\ \ldots$

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$f_{\ell,m}(\theta)$ can be obtained by solving:

$$\mathbb{L}^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$$

$$\left[-\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) - \frac{1}{\sin^2\theta}\partial^2_\varphi\right] f_{\ell m}(\theta)\, e^{im\varphi} = \ell(\ell+1) f_{\ell m}(\theta)\, e^{im\varphi}$$

$$\left[-\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{m^2}{\sin^2\theta}\right] f_{\ell m}(\theta)\, \cancel{e^{im\varphi}} = \ell(\ell+1) f_{\ell m}(\theta)\, \cancel{e^{im\varphi}}$$

(If you want to know $f_{\ell m} = \text{const} \times \frac{1}{\sin^m\theta}\left(\frac{\partial}{\partial\cos\theta}\right)^{\ell-m}(\sin\theta)^{2\ell}$)

NORMALIZATION: $\int Y^*_{\ell m}(\theta\varphi)\, Y_{\ell' m'}(\theta\varphi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}$

$$d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta$$

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2\right)}_{=-\mathbb{L}^2}.$$

SCHRÖDINGER EQ in a central potential:

$$\left[-\frac{1}{2m}\nabla^2 + V(r)\right]\psi(r,\theta,\varphi) = E\,\psi(r,\theta,\varphi).$$

$$\text{set } \psi(r,\theta,\varphi) = R_{E,\ell}(r)\,Y_{\ell,m}(\theta,\varphi)$$

$$-\frac{1}{2m}\cdot\frac{1}{r^2}\partial_r\left(r^2\partial_r R_{E,\ell}\right) + \frac{1}{2m}\cdot\frac{\ell(\ell+1)}{r^2}R_{E,\ell} + V R_{E\ell} = E R_{E,\ell}$$

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Set $R_{E,\ell}(r) = \frac{1}{r} u_{E,\ell}(r)$.

$$-\frac{1}{2m}\frac{d^2}{dr^2} u_{E,\ell} + \Big(\underbrace{\frac{\ell(\ell+1)}{2m\, r^2} + V(r)}_{V_{eff,\ell}(r)}\Big) u_{E,\ell} = E\, u_{E,\ell}$$