Exercise number	Homework	Class exercise	Possible exam type question
1	✓		✓
2			
3		✓	✓
4	✓		
5		✓	✓
6	✓		✓
7			
8	✓		
9		✓	✓
10	✓		✓
11			✓
12		✓	
13		✓	✓
14		✓	✓
15			✓
16		✓	
17		✓	
18	✓		✓
19			
20		✓	
21		✓	
22			✓
23	✓		✓
24		✓	✓
25		✓	

Advanced Analysis exercises

Here is a list of exercises on advanced analysis. Those used as homework are without solution. All others have a solution attached.

Important! Those with a check-mark in the last column are similar to the type of questions that may show up in the exam.

Advanced methods in analysis

1

Use the Kramers-Kronig relations to show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

2

Consider the Schrödinger equation for a free particle on a line, written as

$$L_{x,t}\Psi(x,t) = 0, \quad \text{where} \quad L_{x,t} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + i\hbar\frac{\partial}{\partial t}$$

The retarded Green's function can be written as

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} G(k,t)\, dk$$

Determine $G(k,t)$. (You do not have to compute $G(x,t)$.)

3

Find the retarded Green's function $G(t,t')$ for the damped harmonic oscillator, satisfying

$$\left(m\frac{d^2}{dt^2} + 2mQ^{-1}\omega_0\frac{d}{dt} + m\omega_0^2\right) G(t,t') = \delta(t-t')$$

in the overdamped ($Q < 1$) and underdamped ($Q > 1$) cases. (m is the mass, ω_0 the angular frequency, Q the quality factor of the oscillator.)

Using the above Green's function, write an integral expression for the solution of the driven oscillator with driving force $F(t)$, and show that the solution at time t only depends on the driving force at earlier times.

4

Consider a damped harmonic oscillator with an *extra term* proportional to the *third* time derivative, (τ a constant characteristic time, m is the mass, ω_0 the angular frequency, Q the quality factor of the oscillator.)

$$\left(m\tau\frac{d^3}{dt^3} + m\frac{d^2}{dt^2} + 2mQ^{-1}\omega_0\frac{d}{dt} + m\omega_0^2\right) G(t,t') = \delta(t-t')$$

Show that such a term is not compatible with causality for $Q \to \infty$. To do this, consider a driving force

$$F(t) = \begin{cases} 0, & \text{for } t < 0 \\ F_0, & \text{non-zero constant, for } t \geq 0 \end{cases}$$

and show that the particle starts accelerating before the force is turned on.

This model was originally proposed by Abraham and Lorenz in 1903 to try to describe the energy loss of an accelerating charge due to radiation, but it is clearly faulty. The full resolution of the problem requires quantum mechanics.

5

At the beginning of the 20-th century it was found that the number of nuclear reactions per unit time R in a star was given by

$$R = N \int_0^\infty E \exp\left\{-\left(\frac{E}{k_B T} + \frac{\alpha}{\sqrt{E}}\right)\right\} dE$$

(T is the temperature, E the energy of the collision, k_B the Boltzmann constant, and α, N two other known constants.) Compute this integral in the limit $k_B T \ll \alpha^2$.

6

Consider the Fredholm's integral equation

$$u(x) = \lambda \int_0^1 (1 - 3xy) u(y)\, dy.$$

For what values of λ does it have non-zero solutions? Find those solutions.

7

A membrane is stretched on a drum in the x, y plane, anchored on a curve on this plane. Denote the displacement in the z direction as $u = z(x, y, t)$, a function of the coordinates and time that vanishes at the boundary. (You don't need the explicit expression for the boundary to solve this problem!)

The energy density of the membrane is given by

$$\epsilon = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{s}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)$$

where ρ and s are known constants that depend on the material of which the drum is made.

Find the Lagrangian of the system and derive the equations of motion for u.

8

Check that the Green's function of the one-dimensional heat equation

$$\left(c\frac{\partial}{\partial t} - \mu\frac{\partial^2}{\partial x^2}\right)T(x,t) = \rho(x,t)$$

is given by

$$G(x,t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2\sqrt{\pi\mu c t}}\exp\{-\frac{cx^2}{4\mu t}\} & t > 0 \end{cases}$$

(c is the specific heat, μ the diffusion constant, both assumed known.)

Also show that

$$\lim_{t\to 0^+} G(x,t) = \frac{1}{c}\delta(x)$$

which is interpreted as saying that $G(x,t)$ describes the time evolution of a point-like thermal "spike" at $x = 0$.

Suppose we define the propagation of this thermal front for $t > 0$ by tracking the position of the point $x(t)$ for which, say, $G(x(t),t) = \frac{1}{2}G(0,t)$. (1/2 is just an arbitrary constant.) What is the velocity of the front? Is there anything wrong with it for very small t? How do we explain that?

9

For which real values of λ do the following equations

$$\phi(x) = \lambda\int_{-1}^{1}(t+x)\phi(t)\,dt$$
$$\phi(x) = \lambda\int_{0}^{\pi}\sin(t+x)\phi(t)\,dt$$

have a non-zero solution?

10

Use variational calculus to show the (trivial) result that the shortest path between two points on a plane is a straight line.

11

Consider the integral equation

$$\phi(x) = x + \lambda\int_a^b xt\phi(t)\,dt$$

For what values of λ does the perturbation series converge? Solve the equation exactly. Compare the two solutions. Do you need any condition of λ in the second case?

12

An enclosed area in a field is made of a straight wall of length d and a flexible fence of length $L > d$ attached to the two ends of the wall. Find the equation describing the position of the fence that maximizes the area enclosed.

NB: You do *not* have to solve the equation.

13

Show that the step function

$$\theta(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x > 0 \end{cases}$$

Has the integral representation

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ixt}}{t - i\epsilon} \, dt \qquad (\epsilon > 0)$$

Note that $\theta(0)$ is not well defined by this formula, but it does not really matter, we can define it as we wish (0, 1 or even 1/2). It's just a point and it does not contribute to the values of the integrals in which θ is usually found.

14

Does the equation

$$\phi(x) = x + \int_0^1 (1 + xt)\phi(t) \, dt$$

have a unique solution? Why?

15

Solve the equation

$$\phi(x) = x + \lambda \int_0^1 (xt + t^2)\phi(t) \, dt$$

16

Write the differential equation

$$u''(t) = g(t, u(t)), \quad u(0) = \alpha, u'(0) = \beta$$

as an integral equation.

The (smooth) function g and the initial values α, β are known.

17

A mirage in the desert occurs because light travels faster in hot air than in cold (because the warm air is thinner than the cold). Show using the calculus of variations and Fermat's principle ("a ray of light between two points follows the fastest path") a light beam from a light source at point A with Cartesian coordinates $(x_A, y_A, z_A) = (l, 0, h_A)$ to an observer at the point $B = (x_B, y_B, z_B) = (0, 0, h_B)$ follows a convex curve of the form of an elliptical arc. The desert is assumed here to be completely flat, with l the ground distance between the light source and the observer, and h_A and h_B respectively their height above the ground.

Help: Write the speed of light as $v(h) = v_0 - h/a$, where v_0 is the speed at the ground, h is the height above the ground, and a is a parameter.

18

Compute the following integral in the limit given

$$\int_0^1 e^x \frac{x^n}{(1+x^2)^n}\, dx, \qquad \text{for } n \to \infty$$

19

Consider a rectangular box of width, length and height $W \times L \times H$. The top part of the box is covered by a membrane attached to the edge of the $W \times L$ area. The rest of the walls are rigid. If the box is filled with gas, the membrane stretches so that the interior has volume is $V > W \times L \times H$. Assume V is known. Find (without solving it) the equation that determines the shape of the membrane, under the assumption that it will stretch as to maximize its surface area.

20

A massive rope of length L is hanging with both ends attached to the ceiling spaced by a distance $d < L$. Determine the shape of the rope.

21

Determine the shape of the so-called "Brachistochrone", that is the solution to the following problem posed by Bernoulli in 1696:

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time."

22

Compute the following integral in the limit given

$$\int_0^{\pi} \sqrt{t} e^{x \cos t}\, dt, \qquad \text{for } x \to +\infty$$

23

Compute

$$\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx$$

$$\int_{\mathbf{R}^2} (x^2 y + 1)\, \delta(y - x^2)\, \delta(y - x)\, dx\, dy$$

$$\int_0^{+\infty} dx \int_{-\infty}^{+\infty} dt\, e^{-xt^2}\, \delta(x^2 - a^2)$$

24

Compute

$$\int_{-\pi/2}^{3\pi/2} \delta(\sin x) \cos x\; dx$$

25

Show that

$$G(\mathbf{r}, t) = -\frac{1}{\sqrt{k}(4\pi t)^{3/2}} e^{-r^2/(4kt)} \theta(t)$$

is the Green's function of the 3D heat equation

$$\left(\nabla^2 - \frac{1}{k} \frac{\partial}{\partial t} \right) G(\mathbf{r}, t) = \delta(\mathbf{r}, t)$$

EXERCISE 2.

$$L_{x,t}\, G(x,t) = \delta(x)\delta(t).$$

Fourier Transform in x:

$$\left(\delta(x) = \frac{1}{2\pi}\int dk\, e^{ikx} \right)$$

$$\left(\frac{\hbar^2}{2m}\partial_x^2 + i\hbar\partial_t\right)\frac{1}{2\pi}\int e^{ikx}\, G(k,t)\,dk =$$

$$= \frac{1}{2\pi}\int dk\, e^{ikx}\cdot\delta(t)$$

$$\left(-\frac{\hbar^2}{2m}k^2 + i\hbar\partial_t\right) G(k,t) = 1\cdot\delta(t).$$

Let $G(k,t) = e^{\alpha t}\hat{G}(k,t).$

$$\cancel{-\frac{\hbar^2 k^2}{2m} e^{\alpha t}\hat{G}} + \cancel{i\hbar\alpha e^{\alpha t}\hat{G}} + i\hbar e^{\alpha t}\partial_t\hat{G} = \delta(t)$$

$$\uparrow$$

choose $\alpha = -i\dfrac{\hbar k^2}{2m}$

$$i\hbar\, e^{-i\frac{\hbar k^2}{2m}t}\,\partial_t\hat{G} = \delta(t)$$

$$\partial_t\hat{G} = \frac{i}{\hbar}\, e^{\frac{i\hbar k^2 t}{2m}}\,\delta(t) = \frac{i}{\hbar}\,\delta(t) \quad !$$

$$\Rightarrow \hat{G} = \frac{i}{\hbar}\,\theta(t) \Rightarrow G(k,t) = \frac{i}{\hbar}\, e^{-\frac{i\hbar k^2 t}{2m}}\,\theta(t).$$

EXERCISE 3

We can use time translation invariance to set $t'=0$ (we restore it at the end by letting $t \to t-t'$).

Fourier transform $\left(\delta(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \right)$

$$G(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{G}(\omega).$$

$$\Rightarrow \left(-m\omega^2 - 2im\,Q^{-1}\omega_0 \omega + m\omega_0^2 \right) \tilde{G} = 1$$

$$\tilde{G} = -\frac{1}{m} \frac{1}{\omega^2 + 2iQ^{-1}\omega_0\omega - \omega_0^2}$$

Denominator $= 0$:

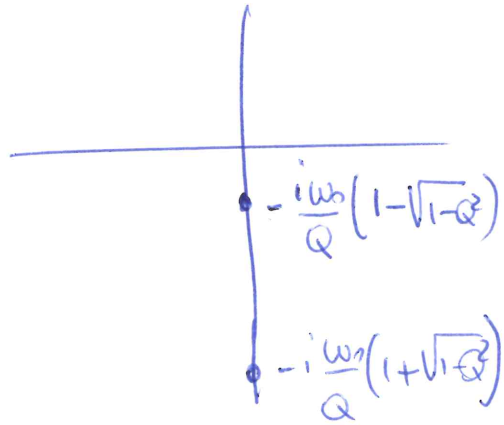
$$\omega_\pm = -i\frac{\omega_0}{Q} \pm \underbrace{\sqrt{-\left(\frac{\omega_0}{Q}\right)^2 + \omega_0^2}}_{\text{argument changes sign at } Q=1}$$

$$\omega_\pm = \begin{cases} \pm\omega_r - i\frac{\omega_0}{Q}, & \omega_r \equiv \omega_0\sqrt{\frac{Q^2-1}{Q^2}} > 0,\ Q>1 \\ \pm i\omega_g - i\frac{\omega_0}{Q}, & \omega_g \equiv \omega_0\sqrt{\frac{1-Q^2}{Q^2}} > 0,\ Q<1. \end{cases}$$

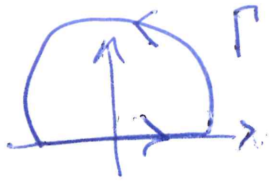
In all cases, the Imaginary parts are below the real axis of $\omega \in \mathbb{C}$

$Q > 1$

$Q < 1$

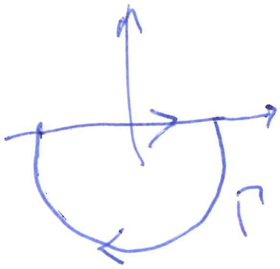


$t < 0$:



$$G(t) = \int_\Gamma \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{G}(\omega) = 0$$

$t > 0$:



$$G(t) = -\int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{G}(\omega) =$$

$$= -\frac{2\pi i}{2\pi}\left(\operatorname*{Res}_{\omega=\omega_+}\left(e^{-i\omega t}\tilde{G}(\omega)\right) + \operatorname*{Res}_{\omega=\omega_-}\left(e^{-i\omega t}\tilde{G}(\omega)\right)\right)$$

$$= -i \cdot \left(-\frac{1}{m}\right)\left(e^{-i\omega_+ t}\frac{1}{\omega_+ - \omega_-} + e^{-i\omega_- t}\frac{1}{\omega_- - \omega_+}\right)$$

$$= \frac{i}{m}\left(e^{-i\omega_+ t} - e^{-i\omega_- t}\right)\frac{1}{2\sqrt{-\left(\frac{\omega_0}{Q}\right)^2 + \omega_0^2}}$$

For the UNDERDAMPED case:

$$= \frac{i}{m} \cdot \frac{1}{2\omega_r} \left(e^{-i\left(\omega_r - i\frac{\omega_0}{Q}\right)t} - e^{-i\left(-\omega_r - i\frac{\omega_0}{Q}\right)t} \right)$$

$$= \frac{1}{m\omega_r} e^{-\frac{\omega_0 t}{Q}} \sin \omega_r t .$$ (Remember: VALID for $t>0$).

Similarly for the OVERDAMPED case

$\omega_r \to i\omega_\rho$ ($\Rightarrow$ sin $\to$ sh).

To get an expression valid $\forall t$ multiply the above by $\Theta(t)$.

Once I have G:

$$x(t) = \int_{-\infty}^{+\infty} G(t-t')F(t')\,dt'$$

If $F(t') = 0$ for $t' < T_{in}$

$$x(t) = \int_{T_{in}}^{+\infty} G(\underbrace{t-t'}_{\text{MUST BE POSITIVE to be non zero!}})F(t')\,dt'$$

$t - t' > 0$, $t' > T_{in} \Rightarrow t > T_{in}$.

EXERCISE 5.

Saddle point method.

write: $\frac{E}{kT} = A x^2$, $\frac{\alpha}{\sqrt{E}} = \frac{A}{x}$.

$$\Rightarrow \frac{\cancel{E}}{kT} \cdot \frac{\alpha^2}{\cancel{E}} = A \cancel{x^2} \cdot \frac{A^2}{\cancel{x^2}} \Rightarrow A = \left(\frac{\alpha^2}{kT}\right)^{\frac{1}{3}} \gg 1$$

$$\Rightarrow x = \frac{1}{\alpha}\left(\frac{\alpha^2}{kT}\right)^{\frac{1}{3}} \sqrt{E}$$

$$\left(E = \alpha^2 x^2 \left(\frac{\alpha^2}{kT}\right)^{-\frac{2}{3}} = \alpha^2 x^2 \cdot A^{-2}\right.$$

$$R = N \cdot \int_0^\infty \alpha^2 A^{-2} x^2 e^{-A\left(x^2 + \frac{1}{x}\right)} \cdot \alpha^2 A^{-2} \cdot 2x\,dx$$

$$= \frac{2N\alpha^4}{A^4} \times \underbrace{\int_0^\infty x^3 e^{-A\left(x^2+\frac{1}{x}\right)} dx}_{\overset{\text{def}}{=} I(A)}.$$

We want to evaluate $I(A)$ for $A \gg 1$.

let $f(x) = x^2 + \frac{1}{x}$.

$f'(x) = 2x - \frac{1}{x^2}$, $f''(x) = 2 + \frac{2}{x^3}$

$f'(x) = 0$ for $x_0^3 = \frac{1}{2}$. there are three

Choose $x_0 = 2^{-\frac{1}{3}}$:

$f(x_0) = 3 \times 2^{-2/3}$, $f'(x_0) = 0$, $f''(x_0) = 6$.

$$I(A) = \int_{-\infty}^{+\infty} x_0^3 \, e^{-A f(x_0) - \frac{A}{2} f''(x_0) \xi^2} d\xi$$

can be extended

$$= x_0^3 e^{-A f(x_0)} \cdot \sqrt{\frac{\pi}{A f''(x_0)/2}} =$$

$$= \frac{1}{2} e^{-3 \cdot 2^{-2/3} A} \cdot \sqrt{\frac{\pi}{3A}}$$

$$\Rightarrow R = \frac{N \alpha^4}{A^4} \cdot \sqrt{\frac{\pi}{3A}} \; e^{-3 \cdot 2^{-2/3} \cdot A}.$$

EXERCISE 7

$$\epsilon = \frac{\rho}{2}(\partial_t u)^2 + \frac{s}{2}\left((\partial_x u)^2 + (\partial_y u)^2\right) = T + V$$

$$L = T - V = \frac{\rho}{2}(\partial_t u)^2 - \frac{s}{2}\left((\partial_x u)^2 + (\partial_y u)^2\right)$$

$$\partial_t \frac{\partial L}{\partial \partial_t u} + \partial_x \frac{\partial L}{\partial \partial_x u} + \partial_y \frac{\partial L}{\partial \partial_y u} =$$

$$= \partial_t \cdot \frac{\rho}{2} \cdot 2\partial_t u - \partial_x \frac{s}{2} 2\partial_x u - \partial_y \frac{s}{2} \cdot 2\partial_y u$$

$$= \rho\,\partial_t^2 u - s(\partial_x^2 + \partial_y^2)u = 0 \,.$$

$$\Rightarrow \underbrace{\frac{\rho}{s}}_{= \frac{1}{c_0^2}} \partial_t^2 u - (\partial_x^2 + \partial_y^2)u = 0$$

$= \frac{1}{c_0^2}$ speed of the wave

$$c_0 = \sqrt{s/\rho}\,.$$

EXERCISE 9

$\boxed{1}$: $\phi(x) = \lambda \int (t+x)\,\phi(t)\,dt$ $\quad \left(\int = \int_{-1}^{+1} \text{ALWAYS} \right)$

$$\phi(x) = \lambda \int t\phi(t)\,dt + \lambda x \int \phi(t)\,dt$$

$$\int \phi(x)\,dx = \lambda \int dx \int t\phi(t)\,dt + \lambda \int x\,dx \int \phi(t)\,dt$$

$$\Rightarrow \quad c_0 = \lambda \cdot 1 \cdot c_1 + \lambda \cdot \frac{1}{2} \cdot c_0$$

$$\int x\phi(x)\,dx = \lambda \int x\,dx \int t\phi(t)\,dt + \lambda \int x^2\,dx \int \phi(x)\,dt$$

$$\Rightarrow \quad c_1 = \lambda \cdot \frac{1}{2} \cdot c_1 + \lambda \cdot \frac{1}{3} \cdot c_0$$

$$\begin{vmatrix} 1 - \frac{1}{2}\lambda & -\lambda \\ -\frac{1}{3}\lambda & 1 - \frac{1}{2}\lambda \end{vmatrix} = 1 - \lambda - \frac{1}{12}\lambda^2 = 0$$

$$\lambda_{\pm} = -6 \pm 4\sqrt{3}$$

$$\phi(x) = \lambda \int \sin(t+x)\,\phi(t)\,dt \qquad \left(\int \equiv \int_0^{\pi} d.. \right).$$

$$\phi(x) = \lambda \cos x \int \sin t\,\phi(t) + \lambda \sin x \int \cos t\,\phi(t).$$

$$\underbrace{\int \sin x\,\phi(x)}_{\equiv S} = \cancel{\lambda \int \overbrace{\cos x \sin x}^{0} \int \sin t\,\phi(t)} + \lambda \int \sin^2 x \overbrace{\int \cos t\,\phi(t)}^{\equiv C}$$

$$\Rightarrow S = 0 + \lambda \cdot \frac{\pi}{2} \cdot C$$

$$\int \cos x\,\phi(x) = \lambda \underbrace{\int \cos^2 x}_{\frac{\pi}{2}} \int \sin t\,\phi(t) + \cancel{\lambda \underbrace{\int \sin x \cos x}_{=0} \int \cos t\,\phi(t)}$$

$$C = \lambda \cdot \frac{\pi}{2} \cdot S + 0$$

$$\Rightarrow \lambda = \frac{2}{\pi}, \quad S = C = (\text{say})\ 1.$$

$$\Rightarrow \phi(x) \propto \sin x + \cos x.$$

EXERCISE 11

$$\phi(x) = x + \lambda \int_a^b x t \, \phi(t) \, dt .$$

Perturbation theory :

$$\phi^{(0)} = x$$

$$\phi^{(1)} = x + \lambda \int_a^b x t^2 dt = x\Big(\underbrace{1 + \lambda \frac{(b-a)^3}{3}}_{= C_1}\Big)$$

$$\phi^{(2)} = x + \lambda \int_a^b x t \cdot C_1 t \, dt =$$

$$= x + \lambda x C_1 \frac{(b-a)^3}{3} = \Big(\underbrace{1 + C_1 \frac{\lambda (b-a)^3}{3}}_{C_2}\Big) x$$

$$\vdots$$

$$C_n = \left(1 + C_{n-1} \frac{\lambda (b-a)^3}{3}\right) = \dots$$

$$= \sum_{k=0}^{n} \left(\lambda \frac{(b-a)^3}{3}\right)^k$$

$\lim_{n \to \infty} C_n$ converges for $\lambda < \dfrac{3}{(b-a)^3}$

Exact solution:

$$\phi(x) = x + \lambda x \int t\phi$$

$$\int x\phi = \int x^2 + \lambda \int x^2 \int t\phi$$

$$\int x\phi = \frac{(b-a)^3}{3} + \lambda \frac{(b-a)^3}{3} \cdot \int t\phi$$

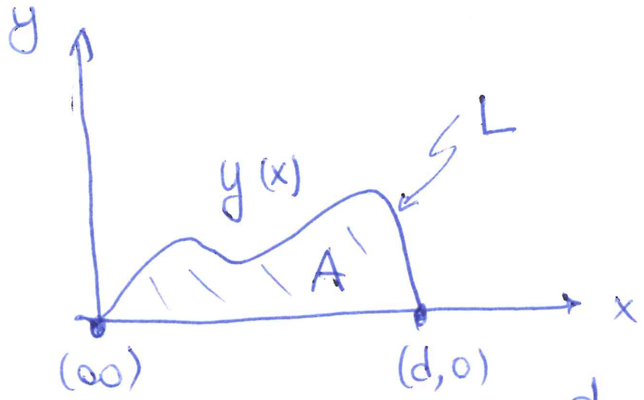
$$\Rightarrow \int x\phi = \frac{(b-a)^3/3}{1 - \lambda \frac{(b-a)^3}{3}}$$

$$\Rightarrow \phi(x) = x + \lambda x \cdot \frac{(b-a)^3/3}{1 - \lambda \frac{(b-a)^3}{3}} =$$

$$= \underbrace{\frac{1}{1 - \lambda \frac{(b-a)^3}{3}}} \cdot x$$

This has the same series expansion as the c_n before for $\lambda < \frac{3}{(b-a)^3}$ but it also makes sense for $\lambda > \frac{3}{(b-a)^3}$, as long as $\lambda \neq \frac{3}{(b-a)^3}$.

EXERCISE 12



$$\hat{A} = \int_0^d y(x)\,dx \quad , \quad \hat{L} = \int_0^d \sqrt{1+y'^2_{(x)}}\,dx$$

Lagrange multiplier:

$$S = \hat{A} + \lambda(\hat{L} - L) = \int_0^d \underbrace{\left(y + \lambda\left(\sqrt{1+y'^2} - L\right)\right)}_{\mathcal{L}} dx$$

$$\pi = \frac{\partial \mathcal{L}}{\partial y'} = \lambda \frac{y'}{\sqrt{1+y'^2}}$$

$$\mathcal{H} = \pi y' - \mathcal{L} = \lambda \frac{y'^2}{\sqrt{1+y'^2}} - y - \lambda\sqrt{1+y'^2} + \lambda L = \text{Const.}$$

absorbe.

$$\Rightarrow \lambda \frac{1}{\sqrt{1+y'^2}} + y = C$$

Solve and fix λ & C with $y(0) = y(d) = 0$.

Not required, but the solution to

$$\begin{cases} \dfrac{\lambda}{\sqrt{1+y'^2}} + y = C \\ y(0) = 0 \quad \text{(ONLY)} \end{cases} \Rightarrow \begin{cases} y' = \sqrt{\dfrac{\lambda^2 - (C-y)^2}{(C-y)^2}} \\ y(0) = 0 \end{cases}$$

is: $y(x) = C \pm \sqrt{C^2 - x^2 \pm 2x\sqrt{\lambda^2 - C^2}}$

To satisfy $y(d) = 0$ as well, one needs to choose the signs:

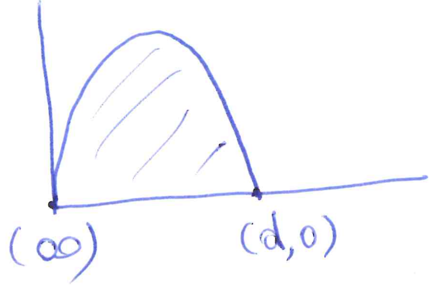
$$y(x) = C + \sqrt{C^2 - x^2 + 2x\sqrt{\lambda^2 - C^2}}$$

with $C < 0$, $2\sqrt{\lambda^2 - C^2} = d$

C is fixed by $L = \int_0^d \sqrt{1+y'^2}\,dx$

$$= \sqrt{4C^2 + d^2}\,\arctan\left(\frac{d}{-2C}\right)$$

looks something like:



EXERCISE 14

$\int = \int_0^1 d..$

$$\phi(x) = x + \int \phi(t) + x \int t\,\phi(t)$$

$$\underbrace{\cancel{\int \phi}}_{C_0} = \frac{1}{2} + 1 \cdot \underbrace{\cancel{\int \phi}}_{C_0} + \frac{1}{2} \underbrace{\int t\,\phi}_{C_1} \Rightarrow C_1 = -1$$

$$\underbrace{\int x\phi}_{C_1} = \frac{1}{3} + \frac{1}{2} \underbrace{\int \phi}_{C_0} + \frac{1}{3} \underbrace{\int t\,\phi}_{C_1} \Rightarrow C_0 = -2$$

$\Rightarrow \phi(x) = x - 2 - x = -2$ is a solution as can be easily checked.

It is unique, since the same computation w/o the $\frac{1}{2}, \frac{1}{3}$ gives $C_0 = C_1 = 0$. No solution to

$$\phi(x) = \int_0^1 (1 + xt)\,\phi(t)\,dt$$

other than $\phi = 0$.

EXERCISE 15

$$\int = \int_0^1 d\ldots$$

$$\phi(x) = x + \lambda x \int t\phi + \lambda \int t^2 \phi$$

$$\underbrace{\int x\phi}_{c_1} = \int x^2 + \lambda \int x^2 \underbrace{\int t\phi}_{c_1} + \lambda \int x \underbrace{\int t^2\phi}_{c_2}$$

$$c_1 = \frac{1}{3} + \frac{\lambda}{3} c_1 + \frac{\lambda}{2} c_2$$

$$\underbrace{\int x^2\phi}_{c_2} = \int x^3 + \lambda \int x^3 \int t\phi + \lambda \int x^2 \int t^2 \phi$$

$$c_2 = \frac{1}{4} + \frac{\lambda}{4} c_1 + \frac{\lambda}{3} c_2$$

$$\underbrace{\begin{pmatrix} 1-\frac{\lambda}{3} & -\frac{\lambda}{2} \\ -\frac{\lambda}{4} & 1-\frac{\lambda}{3} \end{pmatrix}}_{M} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\underbrace{\left(1-\frac{\lambda}{3}\right)^2 - \frac{\lambda^2}{8}}_{\Delta}} \begin{pmatrix} 1-\frac{\lambda}{3} & \frac{\lambda}{2} \\ \frac{\lambda}{4} & 1-\frac{\lambda}{3} \end{pmatrix}$$

invertible for $\lambda \neq -12 \pm 6\sqrt{2}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 1-\frac{\lambda}{3} & \lambda/2 \\ \lambda/4 & 1-\lambda/3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 1/4 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{1}{3} + \frac{1}{72}\lambda \\ \frac{1}{4} \end{pmatrix}$$

$$\Rightarrow \phi = x + \lambda c_1 x + \lambda c_2 =$$

$$= x\left(1 + \frac{\frac{1}{3}\lambda + \frac{\lambda^2}{72}}{\Delta}\right) + \frac{1}{4\Delta}\lambda .$$

EXERCISE 16.

$$\begin{cases} u''(t) = g(t, u(t)) \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

Integrate once:

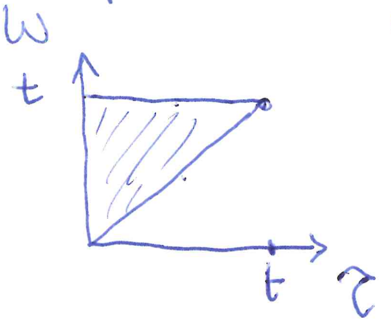
$$\int_0^t u''(\tau)\,d\tau = \int_0^t g(\tau, u(\tau))\,d\tau.$$

$$u'(t) - \beta = \int_0^t g(\tau, u(\tau))\,d\tau$$

Integrate again:

$$\int_0^t u'(\tau)\,d\tau - \beta t = \int_0^t \left(\int_0^w g(\tau, u(\tau))\,d\tau \right) dw$$

$$u(t) - \alpha - \beta t = \int_0^t \left(\int_\tau^t dw \right) g(\tau, u(\tau))\,d\tau$$



$$\Rightarrow \quad u(t) = \alpha + \beta t + \int_0^t (t-\tau)\, g(\tau, u(\tau))\, d\tau$$

Check: $u'(t) = \beta + \cancel{(t-\tau) g(\tau, u(\tau))}\Big|_{\tau = t} +$

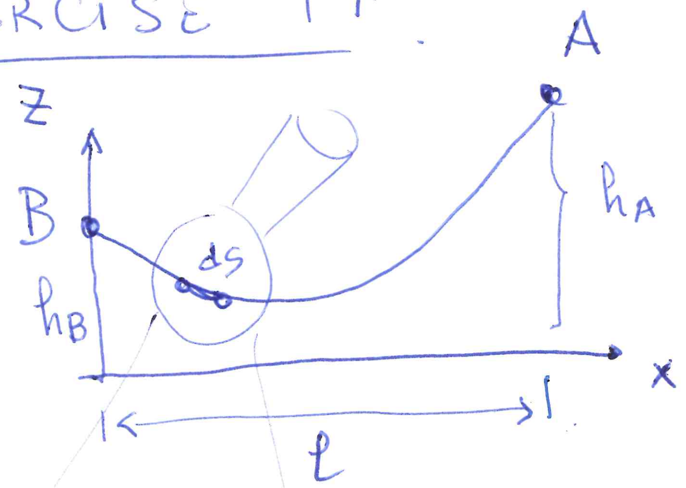
$$+ \int_0^t \frac{d}{dt}\big((t-\tau)\, g(\tau, u(\tau))\big)\, d\tau =$$

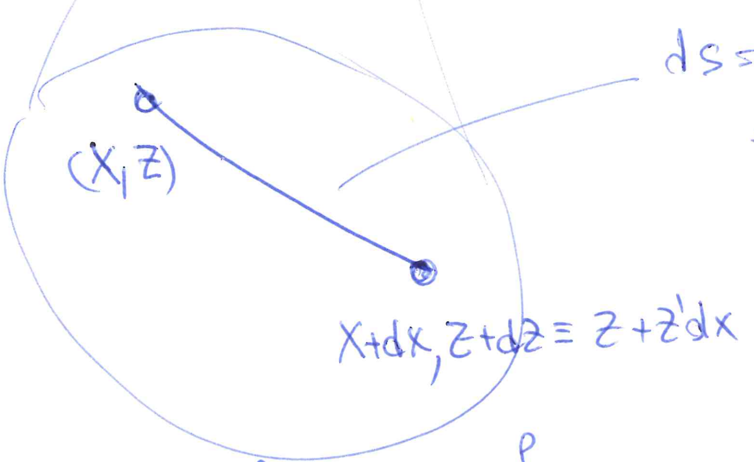
$$= \beta + \int_0^t g(\tau, u(\tau))\, d\tau .$$

$$u''(t) = g(t, u(t)) \quad \checkmark .$$

also $u'(0) = \beta$, $u(0) = \alpha$.

EXERCISE 17





$$ds = \sqrt{dx^2 + z'^2 dx^2}$$
$$= dx\sqrt{1+z'^2}$$

$$T[z] = \int_{x=0}^{x=\ell} \frac{ds}{v} = \int_0^\ell dx \underbrace{\frac{\sqrt{1+z'^2}}{v_0 - z/a}}_{\mathcal{L}}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial z'} = \frac{z'}{\sqrt{1+z'^2}\,(v_0 - z/a)}$$

$$\mathcal{H} = \pi z' - \mathcal{L} = \frac{1}{v_0 - z/a}\left(\frac{z'^2}{\sqrt{1+z'^2}} - \sqrt{1+z'^2}\right) = -\frac{1}{(v_0-\frac{z}{a})\sqrt{1+z'^2}}$$
(const!)

$$\Rightarrow \left(v_0 - \frac{z}{a}\right)\sqrt{1+z'^2} = K$$
const.

$$\Rightarrow z' = \frac{\sqrt{k^2-(av_0-z)^2}}{av_0-z}$$

$$\Rightarrow \int_{h_B}^{h_A} dz \frac{av_0-z}{\sqrt{k^2-(av_0-z)^2}} = \int_0^{\ell} dx = \ell.$$

From which k can be determined.

To find $z(x)$ go back to $z' = \frac{\sqrt{k^2-(av_0-z)^2}}{av_0-z}$

write it as: $(z-av_0)dz + \sqrt{k^2-(z-av_0)^2}\,dx = 0$

and match it with:

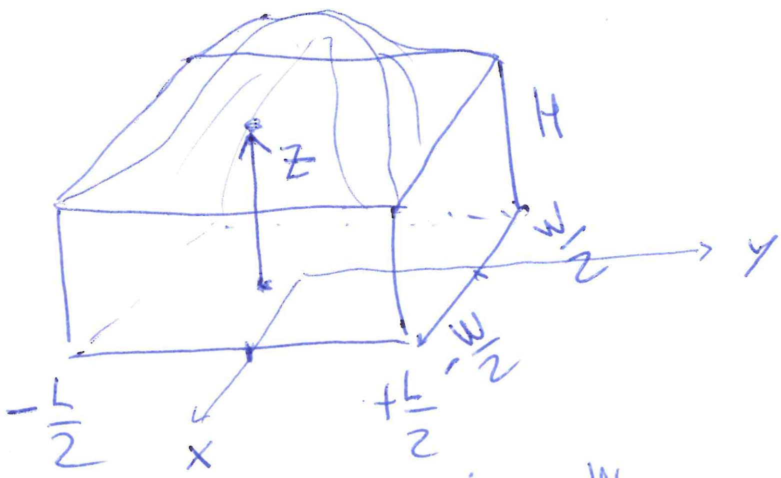
$$(z-av_0)^2 + c_1(x-c_2)^2 = c_3$$

$$\Rightarrow \cancel{2}(z-av_0)dz + \cancel{2}\underbrace{c_1(x-c_2)}_{=\sqrt{k^2-(z-av_0)^2}}dx = 0$$

$$\Rightarrow c_1^2(x-c_2)^2 + (z-av_0)^2 = k^2.$$

c_1 and c_2 fixed by $z(0)=h_B$, $z(\ell)=h_A$.
($c_3 \equiv k^2$ already found.)

EXERCISE 19



$$Vol[z] = \underbrace{H\cdot L\cdot W}_{\substack{\text{const}\\ \text{(drop)}}} + \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \int_{-\frac{W}{2}}^{\frac{W}{2}} dx\, z(x,y)$$

$$= 4 \int_{0}^{L/2} dy \int_{0}^{W/2} dx\, z(x,y)$$

$$Area[z] = 4 \int_{0}^{L/2} dy \int_{0}^{W/2} dx \sqrt{1+(\partial_x z)^2+(\partial_y z)^2}$$

Lagrange: Minimize:

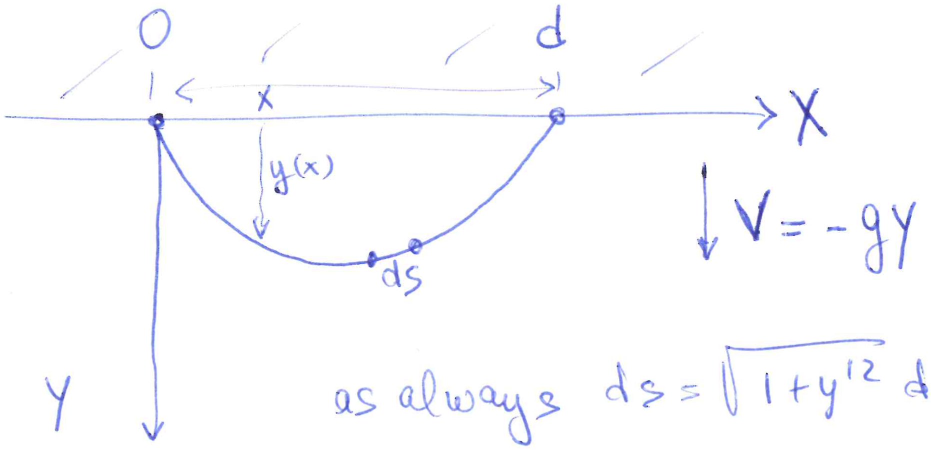
$$\int dx dy \underbrace{\left(\sqrt{1+(\partial_x z)^2+(\partial_y z)^2} - \lambda z\right)}_{\mathcal{L}}$$

$$\Pi_x = \frac{\partial \mathcal{L}}{\partial \partial_x z} = \frac{\partial_x z}{\sqrt{1+(\partial_x z)^2+(\partial_y z)^2}}$$

$$\Pi_y = \frac{\partial \mathcal{L}}{\partial \partial_y z} = \frac{\partial_y z}{\sqrt{1+(\partial_x z)^2+(\partial_y z)^2}}$$

$$\text{E.L:} \quad \partial_x \Pi_x + \partial_y \Pi_y - \underbrace{\frac{\partial \mathcal{L}}{\partial z}}_{=-\lambda} = 0 .$$

EXERCISE 20



as always $ds = \sqrt{1+y'^2}\,dx$

Potential energy:

$$\hat{V}[y] = \int_0^d -\rho g\, y\, ds = -\rho g \int_0^d y\sqrt{1+y'^2}\,dx$$

To be minimized keeping the lenght:

$$\hat{L}[y] = \int_0^d \sqrt{1+y'^2}\,dx = L \quad \text{constant.}$$

$\Rightarrow$ Lagrange multipliers $\quad \hat{S}[y] = \hat{V}[y] - \lambda(\hat{L}[y] - L_0)$

can be dropped.

$$\Rightarrow \hat{S} = \int_0^d \underbrace{(-\rho g y - \lambda)\sqrt{1+y'^2}}_{\mathcal{L}}\,dx.$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial y'} = (-\rho g y - \lambda) \frac{y'}{\sqrt{1+y'^2}} .$$

$$\mathcal{H} = \Pi y' - \mathcal{L} = -(\rho g y + \lambda) \frac{y'^2}{\sqrt{1+y'^2}} + (\rho g y + \lambda)\sqrt{1+y'^2}$$

$$= (\rho g y + \lambda) \frac{1}{\sqrt{1+y'^2}} = (\text{const}) \; C$$

$$\Rightarrow \quad y' = \frac{1}{C} \sqrt{(\rho g y + \lambda)^2 - C^2}$$

$$\Rightarrow \int_0^{y(x)} \frac{C \, dy}{\sqrt{(\rho g y + \lambda)^2 - C^2}} = x$$

$$\Rightarrow \quad y(x) = \frac{C}{\rho g} \cosh\left(\frac{\rho g}{C}(x - C')\right) - \frac{\lambda}{\rho g}$$

λ, C and C' to be fixed by:

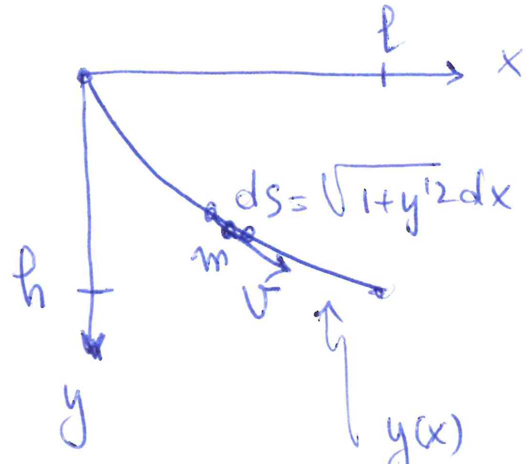
$y(0) = 0 \Rightarrow \quad C \cdot \cosh\left(\rho g \frac{C'}{C}\right) = \lambda$

$y(d) = 0 \Rightarrow \quad C \cdot \cosh\left(\rho g \frac{(d - C')}{C}\right) = \lambda$

$\int_0^d ds = L \Rightarrow \frac{C}{\rho g}\left(\operatorname{sh}\left(\rho g \frac{C'}{C}\right) - \operatorname{sh}\left(\rho g \frac{C' - d}{C}\right)\right) = L$

To be solved numerically

EXERCISE 21



$ds = \sqrt{1+y'^2}\,dx$ as always:

$$\frac{1}{2} m v^2 = m g y$$

$$T[y] = \int_0^{\ell} \frac{ds}{v} = \int_0^{\ell} \underbrace{\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}}_{\mathcal{L}}\, dx$$

$$y' \frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} = \text{Const}$$

$$\Rightarrow \frac{1}{\sqrt{2gy}} \frac{y'^2}{\sqrt{1+y'^2}} - \frac{1}{\sqrt{2gy}} \cdot \frac{1}{\sqrt{1+y'^2}} = \text{Const}$$

$$\Rightarrow y' = \sqrt{\frac{c-y}{y}}$$

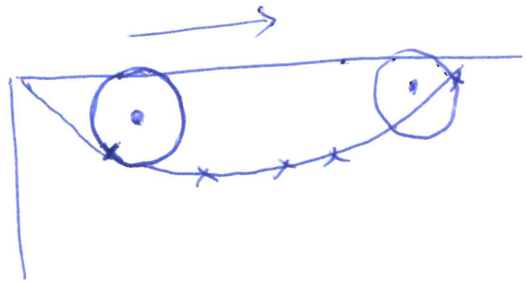
$$\Rightarrow \int_0^{x} dx = \int_0^{y(x)} dy \sqrt{\frac{y}{c-y}}$$

$$\Rightarrow x = -\sqrt{(c-y(x))\,y(x)} + c \,\mathrm{Arcsin}\sqrt{\frac{y(x)}{c}}$$

c fixed by setting $y(\ell) = h$, $x = \ell$.

The curve is actually a cycloid:

$$\begin{cases} x = \frac{K}{2}(\theta - \sin\theta) + K' \\ y = \frac{K}{2}(1 - \cos\theta) \end{cases}$$



EXERCISE 22.

$$I(x) = \int_0^\pi \sqrt{t}\, e^{x\cos t}\, dt$$

within $t \in [0, \pi]$ cos is max at $t = 0$.

$$\cos t \simeq 1 - \frac{t^2}{2}$$

$$I(x) \simeq e^x \int_0^\infty \sqrt{t}\, e^{-\frac{xt^2}{2}}\, dt \quad // u = \frac{xt^2}{2} //$$

$$= e^x \int_0^\infty \left(\frac{2u}{x}\right)^{\frac{1}{4}} \cdot e^{-u} \cdot \frac{du}{x\sqrt{\frac{2u}{x}}} =$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{4}} \cdot \frac{1}{x} \cdot e^x \cdot \Gamma\left(\frac{3}{4}\right).$$

Exercise 24

$$\int_{-\frac{\pi}{2}}^{3\pi/2} \delta(\sin x) \cos x \, dx = I$$

$\sin x = 0$ inside $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$
for $x = 0, \pi$.

$$I = \frac{1}{|\sin' x|} \cos x \Big|_{x=0} + \frac{1}{|\sin' x|} \cos x \Big|_{x=\pi} =$$

$$= 1 + (-1) = 0.$$

EXERCISE 25

Observe: $\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r$ + angular derivatives

$$\Rightarrow \left(\nabla^2 - \frac{1}{k}\frac{\partial}{\partial t}\right) G =$$

$$= \left(\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{1}{k}\partial_t\right)\left(-\frac{1}{\sqrt{k}(4\pi t)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \theta(t)\right) =$$

$$= \theta(t) \underbrace{\left(\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{1}{k}\partial_t\right)\left(-\frac{1}{\sqrt{k}(4\pi t)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}\right)}_{=0 \text{ after some algebra}}$$

$$-\frac{1}{\sqrt{k}(4\pi t)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \cdot \left(-\frac{1}{k}\partial_t \theta\right) =$$

$$= \frac{1}{(4\pi k t)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \delta(t) = \delta_t(\mathbf{r})\, \delta(t).$$

$$= \delta(\mathbf{r})\, \delta(t)$$