Homework 4
Convex Optimization

1. (a) (3 points) Show that $\|\mathbf{x}\|^2$ is smooth with parameter 2.
   (b) (3 points) Let f be a smooth function with parameter L and let $g := f(Ax + b)$. Show that g is smooth with parameter $L \cdot \|A\|^2$ where $\|A\| := \sup_{\|x\|=1} \|Ax\|$ is the *spectral norm* of A.
   (c) (2 points) Deduce that $\|Ax - b\|^2$ is smooth. What is the corresponding parameter?
   (d) (2 points) Give the convergence rate of gradient descent when applied to minimize $\|Ax - b\|^2$ i.e. the number of iterations to decrease the error to ϵ.

2. **Sparse Recovery**. The set-up is as follows. There is an unknown vector $\beta \in R^p$. We get n noisy linear measurements of β: $y_i = x_i^T \beta + e_i, i = 1, ..., n$. We write this in matrix notation: $y = X\beta + e$. Here we have $n < p$. Typically, this means that the problem is under–determined: there are more unknowns than constraints. However, it turns out that if β is *sparse* i.e. has very few non–zero components, then it is possible to solve this. In this problem you will explore solving this problem when β is sparse. Download the given files that will generate the data for three sparse-recovery problems of different sizes. For each problem, we provide: (X, y) — these data specify the problem, and you will use them to compute β. We also provide testing data, (X_{test}, y_{test}). Once you have computed β, you will use the testing data to see how well you did, by computing: $\|X_{test}\beta - y_{test}\|_2$. The files provided also give the parameter λ which is used by the optimization problem specified below as Algorithm 2 (Lasso).

   Use the CVXPY (`https://www.cvxpy.org`) environment.

   (a) (5 points) Algorithm 1: **Least Squares**. Compute a least-squares solution to the problem by solving:

   $$\text{minimize} : \|X\beta - y\|_2.$$

   Figure out how to solve this problem using CVX. Ask CVX to solve each of the three problems, and report: (a) Did CVX succeed? (b) If so, how long did it take to solve each instance? (c) Report the Regression error of the solution computed: $\|X\beta^* - y\|_2$ and also the Testing error: $\|X_{test}\beta - y_{test}\|_2$.

   (b) (5 points) Algorithm 2: **LASSO**.

   $$\text{minimize} : \|X\beta - y\|_2 + \lambda\|\beta\|_1.$$

   Ask CVX to solve each of the three problems, and report: (a) Did CVX succeed? (b) If so, how long did it take to solve each instance? (c) Report the Regression error of the solution computed: $\|X\beta^* - y\|_2$ and also the Testing error: $\|X_{test}\beta - y_{test}\|_2$. (d) What is the support of β i.e. , what are the non-zero coefficients of β?.