Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

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Question 1

(the simplex method)

(2p) a) A non-negative slack (surplus) variable is subtracted from the first constraint to transform the problem into standard form.

$$\begin{array}{lrcrcrcrcl} \text{minimize} & z = 4x_1 & + & 2x_2 & + & x_3, & & & & \\ \text{subject to} & 2x_1 & & & + & x_3 & - & s_1 & = & 3, \\ & 2x_1 & + & 2x_2 & + & x_3 & & & = & 5, \\ & x_1, & & x_2, & & x_3, & & s_1 & \geq & 0. \end{array}$$

We start by formulating a phase 1 problem with an artificial variable $a_1 \geq 0$ added in the first constraint. x_2 can be used as a second basic variable.

$$\begin{array}{lrcrcrcrcrcl} \text{minimize} & w = & & & & & & & & a_1, & & \\ \text{subject to} & 2x_1 & & & + & x_3 & - & s_1 & + & a_1 & = & 3, \\ & 2x_1 & + & 2x_2 & + & x_3 & & & & & = & 5, \\ & x_1, & & x_2, & & x_3, & & s_1, & & a_1 & \geq & 0. \end{array}$$

We start with the BFS given by $(x_2, a_1)^{\mathrm{T}}$. In the first iteration of the simplex algorithm, x_1 has the least reduced cost (-2) and is chosen as the incoming variable. The minimum ration test shows that a_1 should leave the basis. By updating the basis and computing the reduced costs we see that we are now optimal with $w^* = 0$ and we proceed to phase 2.

The BFS is given by $\boldsymbol{x}_B = (x_2, x_1)^{\mathrm{T}}$, $\boldsymbol{x}_N = (x_3, s_1)^{\mathrm{T}}$ and the reduced costs with the phase 2 cost vector are $\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_3,s_1)} = (-\frac{1}{2}, \frac{1}{2})$. The reduced cost for x_3 is negative and x_3 is chosen to enter the basis. $\boldsymbol{B}^{-1}\boldsymbol{b} = (1, \frac{3}{2})^{\mathrm{T}}$ and $\boldsymbol{B}^{-1}\boldsymbol{N}_{x_3} = (0, \frac{1}{2})^{\mathrm{T}}$, therefore x_1 should leave the basis. Updating the basis and computing the new reduced costs gives that $\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,s_1)} = (2, 0) \geq \boldsymbol{0}$ and thus the optimality condition is fulfilled for the current basis. We have $\boldsymbol{x}^*_B = (1, 3)^{\mathrm{T}}$, or in the original variables, $\boldsymbol{x}^* = (x_1, x_2, x_3)^* = (0, 1, 3)^{\mathrm{T}}$, with the optimal value $z^* = 5$.

(1p) b) The marginal improvement (for a non-degenerate optimal solution) when modifying the right-hand-side vector is given by the values of the dual variables (the "shadow prices"). These are given by $\boldsymbol{y}^* = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1} = (0, 1)^{\mathrm{T}}$. Hence, decreasing the first constraint with ϵ gives us nothing. Decreasing the second constraint with ϵ gives an improvement of ϵ of the objective function value.

Question 2

(optimality conditions)

See The Book, Theorem 10.10.

Question 3

(the Frank–Wolfe algorithm)

At $\boldsymbol{x}_0 = (2,1)^{\mathrm{T}}$, $\nabla f(\boldsymbol{x}_0) = (2,1)^{\mathrm{T}}$. Minimizing $2y_1 + y_2$ over the feasible set yields $\boldsymbol{y}_0 = (-1,1)^{\mathrm{T}}$. The search direction therefore is $\boldsymbol{p}_0 = \boldsymbol{y}_0 - \boldsymbol{x}_0 = (-3,0)^{\mathrm{T}}$. The one-dimensional problem (or, line search) then is to minimize $\varphi(\alpha) = f(\boldsymbol{x}_0 + \alpha\boldsymbol{p}_0) = \frac{1}{2}(2-3\alpha)^2 + \frac{1}{2}$, over $\alpha \in [0,1]$. Setting $\varphi'(\alpha) = 0$ yields $\alpha = \frac{2}{3}$; this must be the optimal solution to the line search problem as it belongs to $[0,1]$ and φ is a convex function; the latter holds particularly since f itself is convex. We then obtain, with $\alpha_0 = \frac{2}{3}$, that $\boldsymbol{x}_1 = \boldsymbol{x}_0 + \alpha_0\boldsymbol{p}_0 = (0,1)^{\mathrm{T}}$.

To check whether $\boldsymbol{x}_1$ is optimal, we can, for example, investigate the variational inequality. We have that $\nabla f(\boldsymbol{x}_1) = (0,1)^{\mathrm{T}}$. At $\boldsymbol{x}_1$, all feasible directions are of the form $\{\, \boldsymbol{p} \in \mathbb{R}^2 \mid p_1 \in \mathbb{R},\ p_2 = 0 \,\}$. Hence, for all feasible directions $\boldsymbol{p}$ we have that $\nabla f(\boldsymbol{x}_1)^{\mathrm{T}}\boldsymbol{p} = 0$, and the variational inequality for the problem at hand is fulfilled at $\boldsymbol{x}_1$. Since the problem is convex, $\boldsymbol{x}_1 = (0,1)^{\mathrm{T}}$ must be an optimal solution. [We can also utilize the upper and lower bounds on the optimal value f^* supplied by the algorithm, to reach the same conclusion.]

(3p) Question 4

(modeling)

Introduce the constants d_i for the demand of chocolate in month $i = 1, \ldots, 12$. Let c_1 be the price of 1 kg cocoa from the importer and f_1 the price of 1 kg of cocoa from the market; let c_2 and f_2 be the corresponding prices of sugar. Let a_1 be the amount of cocoa needed for 1 kg of chocolate and a_2 the amount of sugar. Finally let b be the maximal storage capacity.

Introduce the variables x_1 and x_2 for the amount of cocoa/sugar bought from the importer each month. Let y_{1i} and y_{2i} be the amount of cocoa/sugar bought at the local market for months $i = 1, \ldots, 12$. Finally, let z_{1i} and z_{2i} be the amount of cocoa/sugar left in storage after the production in month $i = 1, \ldots, 12$ has been completed. The problem is:

$$\min \quad \sum_{i=1}^{12}\sum_{j=1}^{2} c_j x_j + f_j y_{ji},$$

subject to the constraints

$$\begin{aligned} (x_j + y_{j1} - z_{j1}) &\geq a_j d_1, \quad j = 1,2 \\ (x_j + y_{ji} + z_{ji-1} - z_{ji}) &\geq a_j d_i, \quad i = 2, \ldots, 12, j = 1,2, \\ \sum_{j=1}^{2} z_{ji} &\leq b, \quad i = 1, \ldots, 12, \\ x_j, z_{ji}, y_{ji} &\geq 0 \quad i = 1, \ldots, 12, \quad j = 1,2. \end{aligned}$$

(3p) Question 5

(gradient projection)

Note first that the feasible region X is a circle with center $\boldsymbol{x}_C = (1\ 2)^{\mathrm{T}}$ and radius $r = 1$. Projecting a point $\boldsymbol{y}$ on X results in taking a step of length r in the direction from $\boldsymbol{x}_C$ to $\boldsymbol{y}$. That is:

$$\mathrm{Proj}_X(\boldsymbol{y}) = \begin{cases} \boldsymbol{y} & \text{if } \boldsymbol{y} \in X \\ \boldsymbol{x_C} + r\frac{\boldsymbol{y}-\boldsymbol{x}_C}{\|\boldsymbol{y}-\boldsymbol{x}_c\|} & \text{if } \boldsymbol{y} \notin X \end{cases}$$

The gradient is

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} 2(x_1 + x_2) + 6(x_1 - x_2) \\ 2(x_1 + x_2) - 6(x_1 - x_2) \end{pmatrix}. \tag{1}$$

Iteration 1: $\boldsymbol{x}^0 = (1\ 2)^T$, $\nabla f(\boldsymbol{x}^0) = (0\ 12)^{\mathrm{T}}$. $\boldsymbol{x}^0 - \alpha\nabla f(\boldsymbol{x}^0) = (1\ 2)^{\mathrm{T}} - (0\ 3)^{\mathrm{T}} = (1\ \ -1)^{\mathrm{T}}$. $\mathrm{Proj}_X(1\ \ -1)^{\mathrm{T}} = (1\ 2)^{\mathrm{T}} - (0\ 1)^{\mathrm{T}} = (1\ 1) = \boldsymbol{x}^1$.

Iteration 2: $\boldsymbol{x}^1 = (1\ 1)^T$, $\nabla f(\boldsymbol{x}^1) = (4\ 4)^{\mathrm{T}}$. $\boldsymbol{x}^1 - \alpha\nabla f(\boldsymbol{x}^1) = (1\ 1)^{\mathrm{T}} - (1\ 1)^{\mathrm{T}} = (0\ 0)^{\mathrm{T}}$. $\mathrm{Proj}_X(0\ 0)^{\mathrm{T}} = (1\ 2)^{\mathrm{T}} - \frac{(1\ 2)^{\mathrm{T}}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\binom{\sqrt{5}-1}{2\sqrt{5}-2} = \boldsymbol{x}^2$.

We have convex constraints with an interior point, hence Slaters CQ imply that KKT is necessary for local optimality. The constraint g is active. $\nabla f(\boldsymbol{x}^2) = \binom{0}{12(1-\frac{1}{\sqrt{5}})}$ and $\nabla g(\boldsymbol{x}^2) = \frac{1}{\sqrt{5}}\binom{-2}{-4}$, i.e., they are not parallel. Hence $\boldsymbol{x}^2$ is not a KKT point, and therefore it is not a local (nor a global) minimum.

(3p) Question 6

(a simple optimization problem)

The KKT conditions for this problem amount, apart from complementarity and primal feasibility, to finding a solution in the pair $(\boldsymbol{x}, \mu)^{\mathrm{T}} \in \mathbb{R}^n \times \mathbb{R}_+$ to the nonlinear equations formed by the stationarity conditions for the Lagrangian with respect to $\boldsymbol{x}$, that is, for all $j = 1, \dots, n$,

$$\frac{a_j}{x_j^2} + \frac{\mu}{x_j} = 0.$$

This is clearly impossible, as $x_j > 0$ must be fulfilled, and $a_j > 0$ holds. We therefore conclude that there are not KKT points for this problem.

Can there be optimal solutions that are not KKT points? No, because the linear independence CQ (LICQ) is fulfilled for this problem, so the KKT conditions are necessary conditions for both local and global optimal solutions.

Question 7

(polyhedral theory – LP duality)

Since P is contained in a ball it must be bounded. Also, by assumption it is non-empty. Therefore, for all $\boldsymbol{c} \in \mathbb{R}^n$, there must exist an optimal solution to the problem

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Then, from the Strong duality theorem it is guaranteed that there is an optimal solution also to the problem

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \leq \boldsymbol{0}. \end{aligned}$$

The optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are of course feasible, and we know that $c^{\mathrm{T}}\boldsymbol{x}^* = b^{\mathrm{T}}\boldsymbol{y}^*$ holds (then also $c^{\mathrm{T}}\boldsymbol{x}^* \leq b^{\mathrm{T}}\boldsymbol{y}^*$ holds, which is not true for a general feasible pair). So, with $\boldsymbol{z} = \left((\boldsymbol{x}^*)^{\mathrm{T}}, (\boldsymbol{y}^*)^{\mathrm{T}}\right)^{\mathrm{T}}$ all of the inequalities are fulfilled, and the polyhedron Q is proved to be non-empty.