Chalmers/Gothenburg University
Mathematical Sciences

**EXAM SOLUTION**

# TMA947/MAN280
# APPLIED OPTIMIZATION

**Date:** 11–12–12

**Examiner:** Michael Patriksson

## Question 1

(the simplex method)

**(2p)** a) We first rewrite the problem on standard form. We multiply the objective by $(-1)$ to obtain a minimization problem, multibly the second constraint by $(-1)$ to obtain a positive r.h.s., and introduce slack variables $s_1$ and $s_2$.

$$\begin{array}{llrrrrl} \text{minimize} & z = & -x_1 & -2x_2 & & & \\ \text{subject to} & & x_1 & +x_2 & -s_1 & & = 1 \\ & & -x_1 & +x_2 & & +s_2 & = 2 \\ & & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

In phase I the artificial variable $a$ is added in the first constraint, $s_2$ is used as the second basic variable in order to obtain a unit matrix as the first basis. We obtain the phase I problem

$$\begin{array}{llrrrrrl} \text{minimize} & w = & & & & & a & \\ \text{subject to} & & x_1 & +x_2 & -s_1 & & +a & = 1 \\ & & -x_1 & +x_2 & & +s_2 & & = 2 \\ & & x_1, & x_2, & s_1, & s_2, & a & \geq 0. \end{array}$$

The starting BFS is thus $(a, s_2)^{\mathrm{T}}$. Calculating the vector of reduced costs for the non-basic variables $x_1, x_2, s_1$ yields $(-1, -1, 1)^{\mathrm{T}}$. We can choose between $x_1$ and $x_2$ as entering variable. We let $x_2$ enter the basis. The minimum ratio test shows that $a$ should leave the basis. We thus have a BFS without artificial variables, and may proceed with phase II.

We have the basic variables $(x_2, s_2)$. The vector of reduced costs for the non-basic variables $x_1$ and $s_1$ is $(1, -2)$. We let $s_1$ enter the basis. The minimum ratio test implies that $s_2$ leaves the basis. We now have $x_2, s_1$ as basic variables. The vector of reduced costs for the non-basic variables $x_1$ and $s_1$ is $(-1, 2)^{\mathrm{T}}$. Thus we let $x_1$ enter the basis. We have that the column corresponding to $x_1$ is $\boldsymbol{B}^{-1}\boldsymbol{N}_1 = (-1, -2)^{\mathrm{T}}$. Hence the problem is unbounded.

**(1p)** b) The non-basic variable $s_2 = 0$, as we let $x_1 = \mu$ we have that

$$(x_2, s_1)^{\mathrm{T}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}_1\mu = (2, 1)^{\mathrm{T}} + (1, 2)^{\mathrm{T}}\mu.$$

Returning to the original variables we have that

$$(x_1, x_2)^{\mathrm{T}} = (0, 2)^{\mathrm{T}} + (1, 1)^{\mathrm{T}}\mu$$

is the direction of unboundedness. To see that this is correct draw the problem!

---

## (3p) Question 2

(modeling) Let $x_i$ be the amount of fuel purchased at city $i$, $i = 1, \dots, n$. We also introduce a variable $y_i$ to denote the amount of fuel in the plane when leaving city $i$. Then we can formulate the problem as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} c_i x_i, && (1)\\
\text{subject to} \quad & x_i \leq K_i, \quad i = 1, \dots, n && (2)\\
& z_i - w_i = y_i, \quad i = 1, \dots, n, && (3)\\
& y_i \leq M, \quad i = 1, \dots, n, && (4)\\
& x_i \leq K_i, \quad i = 1, \dots, n, && (5)\\
& y_i \geq \alpha_i z_i, \quad i = 1, \dots, n, && (6)\\
& x_{i+1} + y_i - \alpha_i z_i = y_{i+1}, \quad i = 1, \dots, n-1, && (7)\\
& x_i, y_i, z_i \geq 0, \quad i = 1, \dots, n. && (8)
\end{aligned}
$$

---

## Question 3

(interior penalty methods)

**(1p)** a) All functions involved are in $C^1$. The conditions on the penalty function are fulfilled, since $\phi'(s) = 1/s^2 \geq 0$ for all $s < 0$. Further, LICQ holds everywhere. The answer is yes.

**(2p)** b) With the given data, it is clear that the only constraint is (almost) fulfilled with equality: $(\boldsymbol{x}_6)_1^2 - (\boldsymbol{x}_6)_2 \approx -0.005422 \approx 0$. We set up the KKT conditions to see whether it is fulfilled approximately. Indeed, we have the following corresponding to the system $\nabla f(\boldsymbol{x}_6) + \hat{\mu}_6 \nabla g(\boldsymbol{x}_6) = \mathbf{0}^2$:

$$
\begin{pmatrix} -6.4094265 \\ 3.39524 \end{pmatrix} + 3.385 \begin{pmatrix} 1.88778 \\ -1 \end{pmatrix} \approx \begin{pmatrix} -0.01929 \\ 0.01024 \end{pmatrix},
$$

and the right-hand side can be considered near-zero. Since $\hat{\mu}_6 \geq 0$ we approximately fulfill the KKT conditions.

For the last part, we establish that the problem is convex. The feasible set clearly is convex, since $g$ is a convex function and the constraint is on the "$\leq$"-form. The Hessian matrix of $f$ is

$$
\begin{pmatrix} 12(x_1 - 2)^2 + 2 & -4 \\ -4 & 8 \end{pmatrix},
$$

which is positive semidefinite everywhere (in fact, positive definite outside of the region defined by $x_1 = 2$); hence, $f$ is convex on $\mathbb{R}^2$. We conclude that our problem is convex, and hence the KKT conditions imply global optimality. The vector $\boldsymbol{x}_6$ therefore is an approximate global optimal solution to our problem.

---

## Question 4

(Lagrangian duality)

**(1p)** a) We begin by constructing the Lagrangian function

$$L(\boldsymbol{x}, \boldsymbol{\mu}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}).$$

The dual function is defined as

$$q(\boldsymbol{\mu}) = \min_{x \in \mathbb{R}^n} L(\boldsymbol{x}, \boldsymbol{\mu}).$$

We have that $\nabla_x^2 L(\boldsymbol{x}, \boldsymbol{\mu}) = \boldsymbol{Q}$ which is positive definite, thus the unconstrained problem defining $q$ is convex. We solve the sufficient optimality condition $\nabla_x L(\boldsymbol{x}, \boldsymbol{\mu}) = \boldsymbol{0}$ and obtain

$$\begin{aligned} \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} = \boldsymbol{0}&, \\ \boldsymbol{x} = \boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c})&. \end{aligned}$$

Inserting this into the definition of the Lagrangian function we obtain

$$\begin{aligned} q(\boldsymbol{\mu}) &= \frac{1}{2}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A} - \boldsymbol{c}^{\mathrm{T}})\boldsymbol{Q}^{-1}\boldsymbol{Q}\boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c}) + (\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{\mu}\boldsymbol{A})\boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{b} \\ &= -\frac{1}{2}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A} - \boldsymbol{c})\boldsymbol{Q}^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{c}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{b}. \end{aligned}$$

The dual problem is $\min_{\boldsymbol{\mu} \geq 0} q(\boldsymbol{\mu})$ which is in the same form as the original quadratic program after appropriate restructure of terms.

**(1p)** b) The Hessian of the dual is

$$\nabla^2 q(\boldsymbol{\mu}) = -\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^{\mathrm{T}}.$$

The dual function is always concave, so we know that all eigenvalues are non-negative. The question is if $\boldsymbol{Q}$ has strictly positive eigenvalues, does it imply that the Hessian to $q$ has strictly positive eigenvalues? The answer is no. Consider $\boldsymbol{Q} = \boldsymbol{I}$ and

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

We have that

$$-\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Adding the first rows to the third shows that the rows are linearly dependent, hence $-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ has zero as an eigenvalue. In fact, if $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $m > n$ then we always obtain 0 as an eigenvalue. A simpler counter-example is possible with one variable and two constraints, but one of the constraints will then be redundant.

**(1p)** c) If $\boldsymbol{Q}$ is p.d. then the following holds: Since $\boldsymbol{Q}$ is the Hessian of the primal objective, if $\boldsymbol{Q}$ is p.d. then the primal problem is convex. The dual problem is always a convex problem. The dual function is differentialble since it is a second degree polynomial. For a convex problem, the dual gap is zero.

If $\boldsymbol{Q}$ has a negative eigenvalue then the primal problem is no longer convex. Let $\boldsymbol{v}$ be an eigenvector of $\boldsymbol{Q}$ with negative eigenvalue $\lambda < 0$. We have that

$$L(\alpha\boldsymbol{v}, \boldsymbol{\mu}) = \frac{1}{2}\boldsymbol{\lambda}\alpha^2\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} + \alpha\boldsymbol{c}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{b} - \alpha\boldsymbol{A}\boldsymbol{v}) \to -\infty,$$

as $\alpha \to \infty$. This implies that $q(\boldsymbol{\mu}) := -\infty$ for all $\boldsymbol{\mu}$. Hence the dual gap is no longer zero unless the primal problem is unbounded.

---

## (3p) Question 5

(optimality conditions)

Farkas' Lemma is established in Theorem 11.10.

---

## (3p) Question 6

(LP duality)

We can write the dual problem as

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \geq \boldsymbol{0}^m. \end{aligned}$$

From weak duality, we know that for any primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{y}$, we have $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$. If $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ for a primal feasible $\boldsymbol{x}$ and a dual feasible $\boldsymbol{y}$, we obtain from strong duality that $\boldsymbol{x}$ is optimal in the primal problem, and $\boldsymbol{y}$ is optimal in the dual problem. Hence, all solutions $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$) to the linear inequality system

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} \quad\quad\quad & \geq \boldsymbol{b}, \\ \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} & \leq \boldsymbol{c}, \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} & \leq 0, \\ \boldsymbol{x} \quad\quad\quad & \geq \boldsymbol{0}^n, \\ \boldsymbol{y} & \geq \boldsymbol{0}^m, \end{aligned}$$

will be optimal solutions to the primal (respectively, dual) problem. To find the best optimal solution to the primal problem with respect to the linear function $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}$, we can therefore solve the linear program to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq 0, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \\ & \boldsymbol{y} \geq \boldsymbol{0}^m. \end{aligned}$$

---

## (3p) Question 7

(sequential linear programming)

Suppose that $\boldsymbol{p} = \boldsymbol{0}^n$ solves the SLP subproblem (2). When representing the optimality conditions for this problem, we then note that the bound constraints

(2d) on $\boldsymbol{p}$ are redundant. Writing down the KKT conditions for $\boldsymbol{p}$ in the problem (2), we therefore obtain the conditions that

$$\nabla f(\boldsymbol{x}_k) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}_k) + \sum_{j=1}^{\ell} \lambda_i \nabla h_i(\boldsymbol{x}_k) = \mathbf{0}^n, \tag{1a}$$

$$\mu_i g_i(\boldsymbol{x}^*) = 0, \qquad i = 1, \ldots, m, \tag{1b}$$

$$\boldsymbol{\mu} \geq \mathbf{0}^m. \tag{1c}$$

But this is a statement that $\boldsymbol{x}^*$ is a KKT point in the original problem.