Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

Date: 17–04–12
Examiner: Michael Patriksson

Question 1

(linear programming)

(2p) a) Rewrite the problem into standard form by subtracting slack variables x_5 and x_6 from the left-hand side in the first and second constraint, respectively. If x_2 and x_3 are basic variables, the basic solution is

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 13 \\ 3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus the basic solution is *feasible.*

Now we can check the reduced costs $\bar{\boldsymbol{c}}^T = \boldsymbol{c}_N^T - \boldsymbol{y}^T\boldsymbol{N}$, where $\boldsymbol{y} = \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = \begin{pmatrix} 40 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, for the non-basic variables:

$$\bar{c}_1 = 5 - \begin{pmatrix} 8 & 4 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = 5 \geq 0,$$

$$\bar{c}_4 = -1 - \begin{pmatrix} 8 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = 11 \geq 0,$$

$$\bar{c}_5 = 0 - \begin{pmatrix} 8 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = 8 \geq 0,$$

$$\bar{c}_6 = 0 - \begin{pmatrix} 8 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = 4 \geq 0.$$

All reduced costs are non-negative, and thus the basis is optimal.

(It is also possible to show this using LP duality and complementary slackness conditions.)

(1p) b) The dual solution and the reduced costs are not affected by a small enough perturbation in the right-hand side, and it is therefore enough to study how feasibility is affected.

Basic solution as a function of δ:

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \boldsymbol{B}^{-1}(\boldsymbol{b} - \delta) = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 6-\delta \\ 7-\delta \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 13 \\ 3 \end{pmatrix} - \frac{1}{7}\begin{pmatrix} 2\delta \\ \delta \end{pmatrix}.$$

Constraints on $\delta \geq 0$ for feasibility:

$13 - 2\delta \geq 0 \implies \delta \leq \frac{13}{2}$,
$3 - \delta \geq 0 \implies \delta \leq 3$.

Thus, x_2 and x_3 are optimal basic variables if $0 \leq \delta \leq 3$.

(3p) Question 2

(the Separation Theorem)

See Theorem 4.29 in the course book.

(3p) Question 3

(Lagrangian duality)

Dual problem:

$$\begin{array}{rll} q^* = & \max\limits_{\boldsymbol{\mu} \geq \mathbf{0}} & q(\boldsymbol{\mu}), \\ & \text{where} & q(\boldsymbol{\mu}) = \min\limits_{\boldsymbol{x} \in S} f(\boldsymbol{x}) + \mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}). \end{array}$$

Since the optimal solution to the dual problem is given in the table, it is easy to calculate the dual function $q(\boldsymbol{\mu}^k) = f(\boldsymbol{x}^k) + \mu_1^k g_1(\boldsymbol{x}^k) + \mu_2^k g_2(\boldsymbol{x}^k)$.

Thus, the following calculations can be done:

$q(\boldsymbol{\mu}^1) = -3.0 + 0 \cdot 8.0 + 0 \cdot 12.0 = -3.0,$
$q(\boldsymbol{\mu}^2) = 1.0 - 3 \cdot 3.0 + 3 \cdot 5.0 = 7.0,$
$q(\boldsymbol{\mu}^2) = 9.0 + 1.5 \cdot 2.0 - 6 \cdot 1.0 = 6.0,$
$q(\boldsymbol{\mu}^4) = 12.0 - 2.25 \cdot 1.0 - 4.5 \cdot 0.5 = 7.5,$
$q(\boldsymbol{\mu}^5) = 8.0 + 2 \cdot 0.0 + 3.75 \cdot 1.0 = 11.75,$
$q(\boldsymbol{\mu}^6) = 12.25 - 2.16 \cdot 0.25 - 4 \cdot 0.25 = 10.71.$

Each $q(\boldsymbol{\mu}^k)$ gives an optimistic estimation of the optimal objective function value, f^*. Thus, the best optimistic estimation is $f^* \geq 11.75$.

Every *feasible solution* gives a pessimistic estimation of f^*:

$\boldsymbol{x}^4$ feasible $\implies f^* \leq 12,$
$\boldsymbol{x}^6$ feasible $\implies f^* \leq 12.25.$

Thus, $f^* \leq 12$.

Therefore, the best possible estimation is $11.75 \leq f^* \leq 12$.

(3p) Question 4

(modelling)

To simplify the notations, we change the two dimensions notations into one dimension. So change point (i,j) to $(i-1)\cdot J+j$, and $p_{(i_1,j_1)(i_2,j_2)}$ changes to $p_{(i_1-1)\cdot J+j_1,(i_2-1)\cdot J+j_2}$.

Sets:
$\mathcal{M} := \{i | i \in \{1, ..., I\cdot J\}\}$, the set of possible points,
$\mathcal{N} := \{(i,j)|$ all pairs of points (i,j) where $i \in \mathcal{M}$ is an adjacent point of $j \in \mathcal{M}\}$.

The decision variables are:

$$x_{i,j} = \begin{cases} 1 & \text{part of the optimal route goes from } i \text{ to } j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{i,j\} \in \mathcal{N}$.

Model:

$$\begin{aligned}
\text{maximize} \quad & \prod_{(i,j)\in N} (1 - p_{i,j} x_{i,j}), \\
\text{subject to} \quad & \sum_{j|(i,j)\in N} x_{i,j} = \sum_{k|(k,i)\in N} x_{k,i} && i \in \mathcal{M}\backslash\{1, I\cdot J\}, \\
& \sum_{j|(1,j)\in N} x_{1,j} = \sum_{k|(k,1)\in N} x_{k,1} + 1, \\
& \sum_{j|(I\cdot J,j)\in N} x_{I\cdot J,j} = \sum_{k|(k,I\cdot J)\in N} x_{k,I\cdot J} - 1, \\
& \sum_{(i,j)\in N} x_{i,j} \le S, \\
& x_{i,j} \in \{0,1\} && (i,j) \in \mathcal{N}.
\end{aligned}$$

Question 5

(necessary local and sufficient global optimality conditions)

(1p) a) See Proposition 4.22 in course book.

(2p) b) See Theorem 4.23 in the course book.

Question 6

(true or false)

(1p) a) False. Let $f(x) = -x^2$. At the point $\bar{x} = 0$, all feasible directions $p \neq 0$ are descent directions. However, $f'(\bar{x}) = 0$ and thus $f'(\bar{x})p = 0$. Therefore, the claim is false.

(It is however sufficient, i.e. if $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{p} < 0$, then $\boldsymbol{p}$ is a descent direction with respect to f at $\boldsymbol{x}$.)

(1p) b) False. The problem is feasible but may have an unbounded solution.

(1p) c) False. Consider the function g where $g(x) = 4 - x^2$ and the two points $x^1 = -2$ and $x^2 = 3$ which belong to the set $S = \{x \in \mathbb{R} \mid g(x) \leq 0\}$. By Definitions 3.39 and 3.40, g is concave. However, the point $\frac{1}{2}x^1 + \frac{1}{2}x^2 = \frac{1}{2} \notin S$. Hence, by Definition 3.1, the set S is not convex.

Question 7

(the Karush–Kuhn–Tucker conditions)

(2p) a) First, rewrite the problem to the following form:

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 - x_1, \\
\text{subject to} \quad & 2 - x_1 \leq 0, \\
& (x_1 - 3)^2 - x_2 - 2 \leq 0, \\
& 1 - x_1 + x_2 \leq 0.
\end{aligned}$$

Let:

$g_1(\boldsymbol{x}) = 2 - x_1,$
$g_2(\boldsymbol{x}) = (x_1 - 3)^2 - x_2,$
$g_3(\boldsymbol{x}) = 1 - x_1 + x_2.$

The KKT conditions are:

$$\nabla f(\boldsymbol{x}) + \sum_{i=1}^{3} \mu_i \nabla g_i(\boldsymbol{x}) = \begin{pmatrix} 2x_1 - 1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 2x_1 - 6 \\ -1 \end{pmatrix} + \mu_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$\mu_1, \mu_2, \mu_3 \geq 0,$

$\mu_i g_i(\boldsymbol{x}) = 0, \quad i = 1, 2, 3,$

$g_i(\boldsymbol{x}) \leq 0, \qquad i = 1, 2, 3.$

The following cases of active constraints are possible:

- Let g_1 be active. Solving the KKT conditions gives $x_1 = 2$, $-1 < x_2 < 1$, $\mu_1 = 3$, $\mu_2 = 0$, and $\mu_3 = 0$.
- Let g_1 and g_2 be active. Solving the KKT conditions gives $x_1 = 2$, $x_2 = -1$, $\mu_1 = 3$, $\mu_2 = 0$, $\mu_3 = 0$.
- Let g_2 be active. The KKT conditions do not give any points.
- Let g_2 and g_3 be active. The KKT conditions do not give any points.
- Let g_3 be active. The KKT conditions do not give any points.
- Let $g_!$ and g_3 be active. Solving the KKT conditions gives $x_1 = 2$, $x_2 = 1$, $\mu_1 = 3$, $\mu_2 = 0$, $\mu_3 = 0$.
- Let no constraints be active. The KKT conditions do not give any points.

Thus, the feasible points fulfilling the KKT conditions are $\boldsymbol{x} = \begin{pmatrix} 2 \\ a \end{pmatrix}$, where $-1 \leq a \leq 1$.

(1p) b) The objective function f and the constraint functions g_i are convex. Therefore the KKT conditions are sufficient for global optimality, and thus all KKT points are optimal.