Lecture 10

LP duality

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Consider the **primal LP** written in standard form:

$$\begin{aligned} z^* = \text{infimum} \quad & \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{P}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$. The corresponding **dual LP** is

$$\begin{aligned} q^* = \text{supremum} \quad & \boldsymbol{b}^T \boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \in \mathbb{R}^m. \end{aligned} \tag{D}$$

(P) Minimization problem with n variables and m constraints.

(D) Maximization problem with m variables and n constraints.

Simplex algorithm, review

- At a **basic feasible solution** (BFS), the variables can be ordered s.t.

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix}, \quad A = (\boldsymbol{B}, \boldsymbol{N}), \quad \boldsymbol{c} = \begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{c}_N \end{pmatrix},$$

  where $\boldsymbol{x}_B$ are **basic variables** and $\boldsymbol{x}_N$ the **non-basic variables**.

- For a specific **basis matrix** $\boldsymbol{B}$, we have that

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b},$$
$$\boldsymbol{x}_N = \mathbf{0}^{n-m}$$

- Simplex algorithm iteratively updates $\boldsymbol{B}$, one column at a time, until it terminates (optimality or objective value $\to -\infty$).

Introducing a dual vector

- Apply simplex algorithm, and assume an optimal basis B is found
- Optimal basis B means that the reduced costs are nonnegative:

$$\tilde{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N} \geq (\mathbf{0}^{n-m})^T \tag{1}$$

We introduce the **(optimal dual) vector**

$$\boldsymbol{y}^* := (\boldsymbol{c}_{\mathbf{B}}^{\mathbf{T}} \boldsymbol{B}^{-\mathbf{1}})^{\mathbf{T}} \tag{2}$$

- By definition in (2), $b^T y^* = (y^*)^T b = c_B^T (B^{-1} b) = c_B^T x_B = c^T x^*$
- In addition, by optimality (i.e., (1)),

$$\left.\begin{array}{l} \boldsymbol{c}_B^T - (\boldsymbol{y}^*)^T \boldsymbol{B} = \mathbf{0}^m \\ \boldsymbol{c}_N^T - (\boldsymbol{y}^*)^T \boldsymbol{N} \geq (\mathbf{0}^{n-m})^T \end{array}\right\} \Longrightarrow \boldsymbol{c}^T - (\boldsymbol{y}^*)^T \boldsymbol{A} \geq \mathbf{0}^n$$

Thus, y^* **satisfies** $\boldsymbol{A}^T \boldsymbol{y}^* \leq \boldsymbol{c}$ **and** $b^T y^* = c^T x^*$

Dual vector in dual LP

- $\boldsymbol{y}^* := (\boldsymbol{c}_B^T \boldsymbol{B}^{-1})^T$ is feasible to dual problem (D)

$$\begin{aligned} q^* = \text{supremum} \quad & \boldsymbol{b}^T \boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^T \boldsymbol{y} \le \boldsymbol{c}, \\ & \boldsymbol{y} \in \mathbb{R}^m. \end{aligned} \tag{D}$$

- AND, $\boldsymbol{y}^*$ achieve $b^T y^* = c^T x^* = z^* =$ primal optimal obj. value

- As we will see, y^* is indeed optimal to (D). Why?

Weak duality

For any $\boldsymbol{x} \in \mathbb{R}^n$ feasible to (P), and $\boldsymbol{y} \in \mathbb{R}^m$ feasible to (D):

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n \end{aligned} \qquad \text{and} \qquad \begin{aligned} \boldsymbol{A}^T\boldsymbol{y} &\leq \boldsymbol{c}, \\ \boldsymbol{y} &\in \mathbb{R}^m \end{aligned},$$

we have

$$\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{y} \implies \boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{b}^T\boldsymbol{y}$$

- $z^* \geq q^*$ (i.e., optimal primal obj. val $\geq$ optimal dual obj. val)
- We maximize $\boldsymbol{b}^T\boldsymbol{y}$ (s.t. $A^Ty \leq c$) to get the best lower bound.

The dual LP

Dual problem = problem to find best primal objective lower bound

$$\begin{aligned} q^* = \text{supremum} \quad & \boldsymbol{b}^T \boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \in \mathbb{R}^m. \end{aligned} \tag{D}$$

- Under simple assumption, $q^* = z^*$ (remember $y^* = (c_B^T B^{-1})^T$?)
- (D) = **Lagrangian dual problem** "dualizing" $Ax = b$ in (P)
- Can define dual problems for all types of LPs (not just standard form)

Dual problem, general construction

- Let A be the constraint matrix. Let a_i^T denote the i-th row of A, and A_j denote the j-th column of A:

(primal problem)

$$
\begin{array}{lll}
\underset{x}{\text{minimize}} & c^T x & \\
\text{subject to} & a_i^T x \geq b_i, & i \in M_1, \\
& a_i^T x \leq b_i, & i \in M_2, \\
& a_i^T x = b_i, & i \in M_3, \\
& x_j \geq 0, & j \in N_1 \\
& x_j \leq 0, & j \in N_2 \\
& x_j \in \mathbb{R}^n, & j \in N_3
\end{array}
$$

(dual problem)

$$
\begin{array}{lll}
\underset{y}{\text{maximize}} & b^T y & \\
\text{subject to} & y_i \geq 0, & i \in M_1, \\
& y_i \leq 0, & i \in M_2, \\
& y_i \in \mathbb{R}^m, & i \in M_3, \\
& A_j^T y \leq c_j, & j \in N_1, \\
& A_j^T y \geq c_j, & j \in N_2, \\
& A_j^T y = c_j, & j \in N_3.
\end{array}
$$

Duality

- In class, we discuss properties of pair (P) and (D)

$$\begin{aligned} z^* = \text{infimum} \quad & \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \tag{P}$$

$$\begin{aligned} q^* = \text{supremum} \quad & \boldsymbol{b}^T \boldsymbol{y}, \\ \text{subject to} \quad & \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \in \mathbb{R}^m. \end{aligned} \tag{D}$$

- Analogous properties hold for other types of LP primal and dual

Weak duality restated

Weak duality theorem

If $\boldsymbol{x}$ is a feasible solution to (P) and $\boldsymbol{y}$ is a feasible solution to (D), then

$$\boldsymbol{c}^T \boldsymbol{x} \geq \boldsymbol{b}^T \boldsymbol{y}.$$

Proof: See slide 6

Weak duality, unboundedness

Corollary

- If the optimal objective value of primal problem (P) is $-\infty$, then dual problem (D) is infeasible.
- If the optimal objective value of dual problem (D) is $+\infty$, then primal problem (P) is infeasible.

Proof: Show first statement by contrapositive. Suppose $y \in \mathbb{R}^m$ feasible to (D). Then, $z^* \geq b^T y > -\infty$ by weak duality. Second statement similar (home exercise).

Corollary

If $\boldsymbol{x}$ is a feasible solution to (P), $\boldsymbol{y}$ is a feasible solution to (D), and

$$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y},$$

then $\boldsymbol{x}$ is optimal in (P) and $\boldsymbol{y}$ is optimal in (D).

Proof: By statement assumption and weak duality theorem,

$$\begin{aligned} c^T x = b^T y \leq c^T \tilde{x}, &\quad \forall \tilde{x} : A\tilde{x} = b,\ \tilde{x} \geq \mathbf{0}, \\ b^T y = c^T x \geq b^T \tilde{y}, &\quad \forall \tilde{y} : A^T \tilde{y} \leq c \end{aligned}$$

Thus, $\boldsymbol{x}$ is optimal in (P) and $\boldsymbol{y}$ is optimal in (D).

Strong duality theorem

If both (P) and (D) are feasible, then

1. There exist x^* optimal to (P) and y^* optimal to (D)
2. $c^T x^* = b^T y^*$ (so, $z^* = q^*$)

Proof:

- (P) and (D) feasible implies simplex algorithm terminates with an optimal basis matrix B associated with optimal x^* (why?)
- Construct $(y^*) = \left(c_B^T B^{-1}\right)^T$, then $A^T y^* \leq c$ and $c^T x^* = b^T y^*$ (see Slide 4). Then y^* is optimal to (D) (why?)
- Hint: "why?" $=$ weak duality

Minimum cut problem:
What is minimum rail capacities (z^*) to destroy to prevent the Soviets from sending troops, in the event of war?

$$
\begin{aligned}
z^* = \min_{p,q} \;& u^T q \\
\text{s.t. } & A^T p + q \geq 0 \\
& b^T p \geq 1 \\
& q \geq 0
\end{aligned}
$$

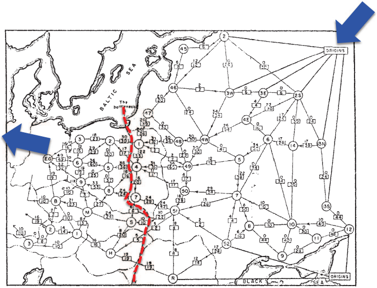

Figure: Western USSR railroad

Maximum flow problem:

What is maximum number of troops (q^*) the Soviets can send?

$$
\begin{aligned}
q^* = \max_{x,s} \quad & s \\
\text{s.t.} \quad & Ax + bs = 0 \\
& x \leq u \\
& x, s \geq 0
\end{aligned}
$$

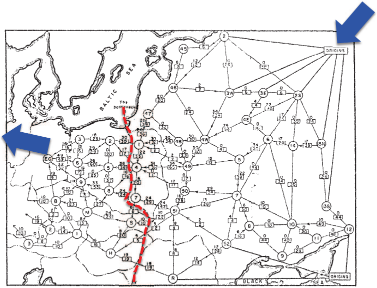

Figure: Western USSR railroad

LP, matrix of possibilities

- For a LP, only three possibilities are allowed:
  1. There is a finite optimal solution.
  2. The optimal objective value is unbounded (e.g., $-\infty$ for minimization problem).
  3. The problem is infeasible.

- A LP and its dual can have the following possibilities:

(D)\(P)	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

Complementary slackness, I

Complementary Slackness Theorem

Let $\boldsymbol{x}$ be feasible in (P) and $\boldsymbol{y}$ feasible in (D). Then

$$\left.\begin{array}{r}\boldsymbol{x} \text{ optimal to (P)} \\ \boldsymbol{y} \text{ optimal to (D)}\end{array}\right\} \iff x_j(c_j - \boldsymbol{A}_{\cdot j}^T \boldsymbol{y}) = 0, \quad j = 1, \dots, n,$$

where $\boldsymbol{A}_{\cdot j}$ is column j of $\boldsymbol{A}$.

Proof:

$$c^T x = b^T y \overset{Ax=b}{\iff} (c^T x - y^T A x) = 0 \;\underset{A^T y \leq c}{\overset{x \geq \mathbf{0},\ Ax=b}{\iff}}\; x_j(c_j - \boldsymbol{A}_{\cdot j}^T \boldsymbol{y}) = 0,\ \forall j$$

Complementary slackness "$\Longrightarrow$" direction due to strong duality
Complementary slackness "$\Longleftarrow$" direction due to weak duality

Complementary slackness II

Complementary Slackness Theorem

Let $\boldsymbol{x}$ be feasible in (P) and $\boldsymbol{y}$ feasible in (D). Then

$$\left.\begin{array}{l}\boldsymbol{x} \text{ optimal to (P)} \\ \boldsymbol{y} \text{ optimal to (D)}\end{array}\right\} \iff x_j(c_j - \boldsymbol{A}_{\cdot j}^T \boldsymbol{y}) = 0, \quad j = 1, \ldots, n,$$

where $\boldsymbol{A}_{\cdot j}$ is column j of $\boldsymbol{A}$.

For a primal-dual pair of optimal solutions $\boldsymbol{x}^*$, $\boldsymbol{y}^*$

- If there is slack in one constraint, then the respective variable in the other problem is zero.
- If a variable is positive, then there is no slack in the respective constraint in the other problem.

Complementary slackness, III

Consider primal and dual pair

$$
\text{(P')} \quad \begin{aligned} \underset{x}{\text{maximize}} \quad & c^T x \\ \text{subject to} \quad & Ax \le b \\ & x \ge \mathbf{0} \end{aligned}
\qquad\qquad
\text{(D')} \quad \begin{aligned} \underset{y}{\text{minimize}} \quad & b^T y \\ \text{subject to} \quad & A^T y \ge c \\ & y \ge \mathbf{0} \end{aligned}
$$

Complementary Slackness Theorem Let x feasible to (P') and y feasible to (D'). Then,

$$
\begin{cases} \boldsymbol{x} \text{ optimal to (P')} \\ \boldsymbol{y} \text{ optimal to (D')} \end{cases} \iff \begin{cases} x_j(c_j - y^T A_{\cdot j}) = 0, & j = 1, \dots, n \\ y_i(A_{i\cdot} x - b_i) = 0, & i = 1, \dots, m \end{cases}
$$

Perturbing equality constraint

Let B be optimal basis of

$$(P): \quad \begin{aligned} v(b) := \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq \mathbf{0}. \end{aligned}$$

Consider problem with perturbed equality constraint RHS b':

$$(P'): \quad \begin{aligned} v(b') := \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b' (= b + \Delta b), \\ & x \geq \mathbf{0}. \end{aligned}$$

Small constraint perturbation

- Suppose in (P) $x_B = B^{-1}b > 0$ (i.e., **non-degenerate**). Then,

$$|\Delta b| \text{ small enough} \implies x'_B = B^{-1}b' = x_B + B^{-1}\Delta b \geq 0$$
$$B \text{ opt in } (P) \implies \tilde{c}_N = (c_N^T - c_B^T B^{-1} N)^T \geq 0 \implies B \text{ opt in } (P')$$

- Perturbed optimal objective value **locally linear** near $b' = b$

$$v(b') = c_B^T x'_B = c_B^T B^{-1} b' = (y^*)^T b'$$

Shadow price theorem

If, for a given vector $\boldsymbol{b}$, the optimal BFS of (P) is non-degenerate, then its optimal objective value is differentiable at $\boldsymbol{b}$, with

$$\nabla v(\boldsymbol{b}) = \boldsymbol{y}^*$$

Consider LP

$$\begin{array}{lrcrcrcrl} \text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & \\ \text{subject to} & x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = 2 \\ & -2x_1 & + & x_2 & + & & + & x_4 & = 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq 0 \end{array}$$

- Optimal basis $B = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$, $x^* = (0, 3, 2, 0)^T$
- optimal objective value $v(b) = 5$
- Optimal dual vector $(y^*)^T = c_B^T B^{-1} = (-\frac{1}{2}, 2)$

Shadow price, illustration

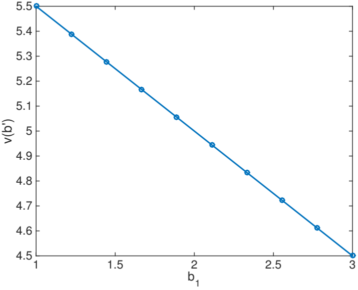

Figure: $v(b + \Delta b_1 e_1)$ vs Δb_1

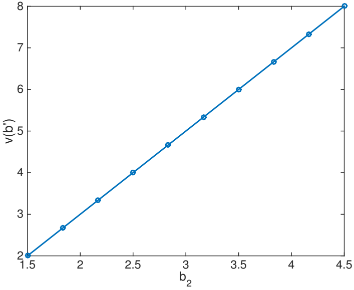

Figure: $v(b + \Delta b_2 e_2)$ vs Δb_2

Consistent with shadow price theorem: $\nabla v(b) = y^* = (-\frac{1}{2}, 2)^T$

Large constraint perturbation

- Consider perturbed LP, with Δb large

$$(P(b')): \quad \begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b'(= b + \Delta b), \\ & x \geq \mathbf{0}. \end{aligned}$$

- Let B be optimal basis of $P(b)$. $\bar{x} = \begin{pmatrix} \boldsymbol{B}^{-1} b' \\ \mathbf{0}^{n-m} \end{pmatrix}$ satisfies
  - basic solution to $P(b')$
  - not necessarily feasible to $P(b')$ when Δb large
  - nonnegative reduced costs: $\tilde{c}_N = (c_N^T - c_B^T B^{-1} N)^T \geq 0$
- B initial basis for **dual simplex method** to solve $P(b')$

Perturbations in the objective

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & (\boldsymbol{c} + \Delta c)^T \boldsymbol{x}, \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\
& \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
\qquad (P(\Delta c))
$$

- $B =$ optimal basis for $P(0)$;
- $\bar{x} = \begin{pmatrix} \boldsymbol{B}^{-1} b \\ \boldsymbol{0}^{n-m} \end{pmatrix}$ BFS in $P(\Delta c)$, but optimal?
- Sufficient condition for optimality:

$$
\tilde{\boldsymbol{c}}_N^T = (\boldsymbol{c}_N + \Delta c_N)^T - (\boldsymbol{c}_B + \Delta c_B)^T \boldsymbol{B}^{-1} \boldsymbol{N} \geq \boldsymbol{0}
$$

- If only one **non-basic** component of $\boldsymbol{c}_N$ is perturbed, i.e.,

$$\Delta c = \begin{pmatrix} \Delta c_B \\ \Delta c_N \end{pmatrix} = \begin{pmatrix} \mathbf{0}^m \\ \varepsilon \boldsymbol{e}_j \end{pmatrix}, \qquad \varepsilon \in \mathbb{R}$$

- B optimal in the perturbed problem $P(\Delta c)$ if

$$\begin{aligned} \tilde{\boldsymbol{c}}_N^T &= (\boldsymbol{c}_N + \Delta c_N)^T - (\boldsymbol{c}_B + \Delta c_B)^T \boldsymbol{B}^{-1} \boldsymbol{N} \\ &= (\boldsymbol{c}_N + \varepsilon \boldsymbol{e}_j)^T - (\boldsymbol{c}_B)^T \boldsymbol{B}^{-1} \boldsymbol{N} \\ &= \underbrace{c_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}}_{\text{unperturbed}} + \varepsilon \boldsymbol{e}_j^T \geq \mathbf{0} \end{aligned}$$

- Need to check **only one entry**:

$$(\tilde{\boldsymbol{c}}_N)_j = (\boldsymbol{c}_N)_j + \varepsilon - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}_j \geq 0$$

- If only one **basic** component of $\boldsymbol{c}_B$ is perturbed, i.e.,

$$\Delta c = \begin{pmatrix} \Delta c_B \\ \Delta c_N \end{pmatrix} = \begin{pmatrix} \varepsilon \boldsymbol{e}_j \\ \boldsymbol{0}^{n-m} \end{pmatrix} \qquad \varepsilon \in \mathbb{R}$$

- B optimal in the perturbed problem $P(\Delta c)$ if

$$\begin{aligned} \tilde{\boldsymbol{c}}_N^T &= (\boldsymbol{c}_N + \Delta c_N)^T - (\boldsymbol{c}_B + \Delta c_B)^T \boldsymbol{B}^{-1}\boldsymbol{N} \\ &= \boldsymbol{c}_N^T - (\boldsymbol{c}_B + \varepsilon \boldsymbol{e}_j)^T \boldsymbol{B}^{-1}\boldsymbol{N} \\ &= \underbrace{\boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{N}}_{\text{unperturbed}} - \varepsilon e_j^T \boldsymbol{B}^{-1}\boldsymbol{N} \geq \boldsymbol{0} \end{aligned}$$

- $\varepsilon e_j^T \boldsymbol{B}^{-1}\boldsymbol{N}$ is dense; need to check the **whole vector** for $\tilde{\boldsymbol{c}}_N \geq \boldsymbol{0}$

Large perturbations in the objective

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & (\boldsymbol{c} + \Delta c)^T \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned} \tag{$P(\Delta c)$}$$

- $B =$ optimal basis for $P(0)$;
- Δc large, sufficient condition for optimality need not hold

$$\tilde{\boldsymbol{c}}_N^T = (\boldsymbol{c}_N + \Delta c_N)^T - (\boldsymbol{c}_B + \Delta c_B)^T \boldsymbol{B}^{-1}\boldsymbol{N} \not\geq \mathbf{0}$$

- B corresponds to BFS in $(P(\Delta c)$; start simplex method with B