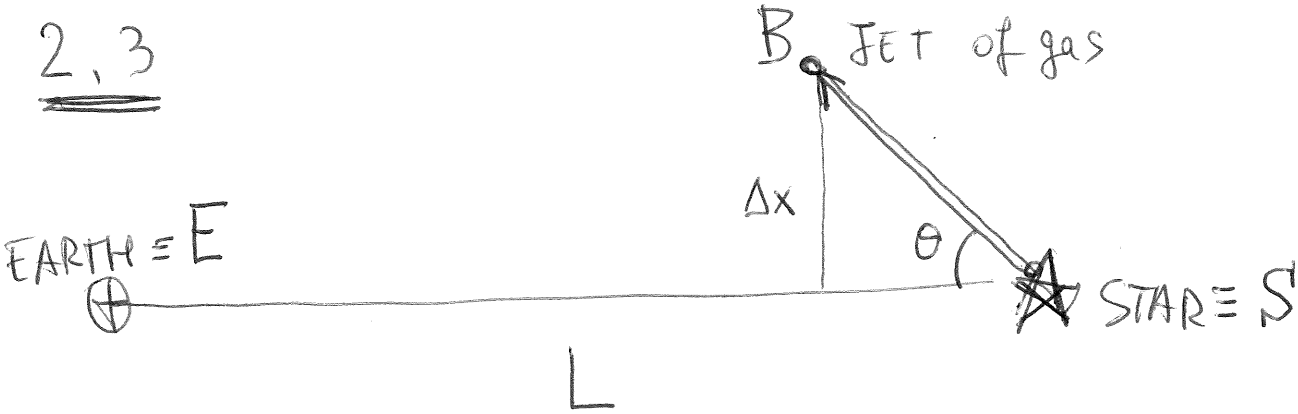
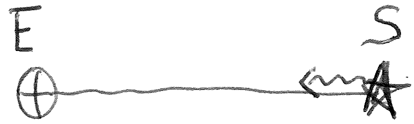
2.3

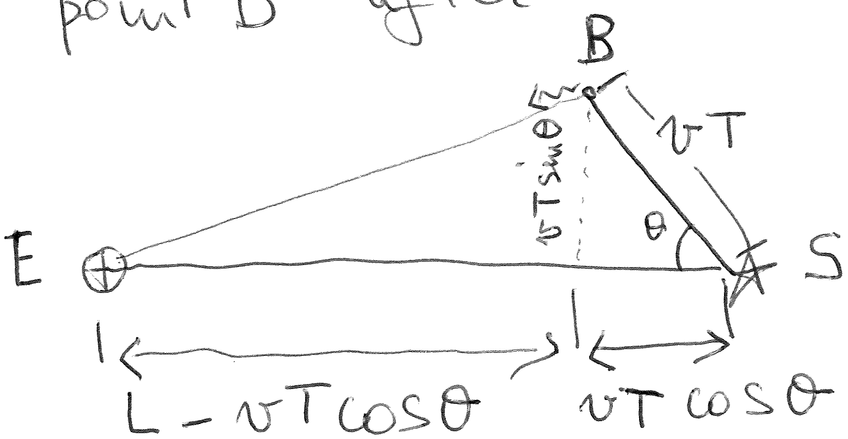


Suppose the jet is emitted at $t=0$ in the Earth frame:

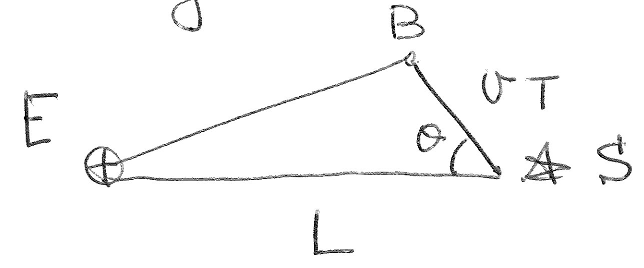


Light from the emission reaches the Earth after a time $t_1 = L/c$.

Suppose the jet reaches some point B after a time T



Using the cos- theorem:



$$EB = \sqrt{L^2 + v^2T^2 - 2LvT\cos\theta}$$

you can do the calculation exactly but you can make your life much easier by noticing that

$L \gg vT$ (The distance to the star is $\gg$ the lenght of the jet).

$$\Rightarrow EB \simeq L - vT\cos\theta$$

So, the time it takes for the light from B to reach the Earth:

$$t_2 \cong T + \frac{L - vT\cos\theta}{c}$$

Now if you just think (wrongly) at the jet as emitted $\perp$ the the ES, you estimate the transverse velocity as:

$$V_\perp^{\text{apperent}} = \frac{\Delta x}{\Delta t} = \frac{vT\sin\theta}{t_2 - t_1} =$$

$$= \frac{vT\sin\theta}{T - \frac{v}{c}T\cos\theta} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}$$

that CAN BE $> c$ (!) for $v<c$:

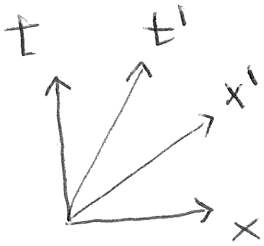
Limit case

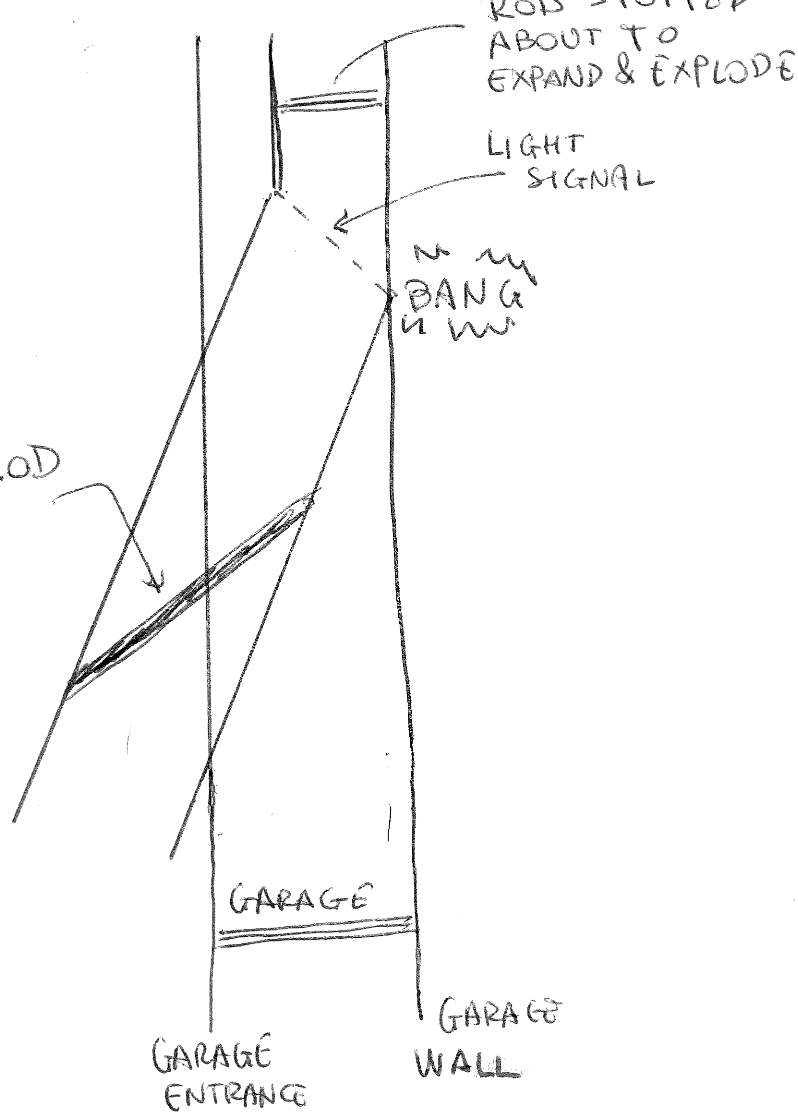
$$c \overset{\downarrow}{=} \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} \Rightarrow v = \frac{c}{\sin\theta + \cos\theta}$$

note that $\sin\theta + \cos\theta$ is always ≥ 1 for $\theta \in [0, \pi/2]$.

3.3

From the garage frame:





see Ladder Paradox in Wikipedia.

3.5

a) $T = \gamma \tau = \frac{1}{\sqrt{1 - v^2/c^2}} \times \tau =$

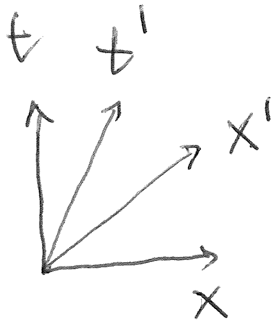
$$= \frac{26\,ms}{\sqrt{1 - 0.95^2}} = 83.3\,ms$$

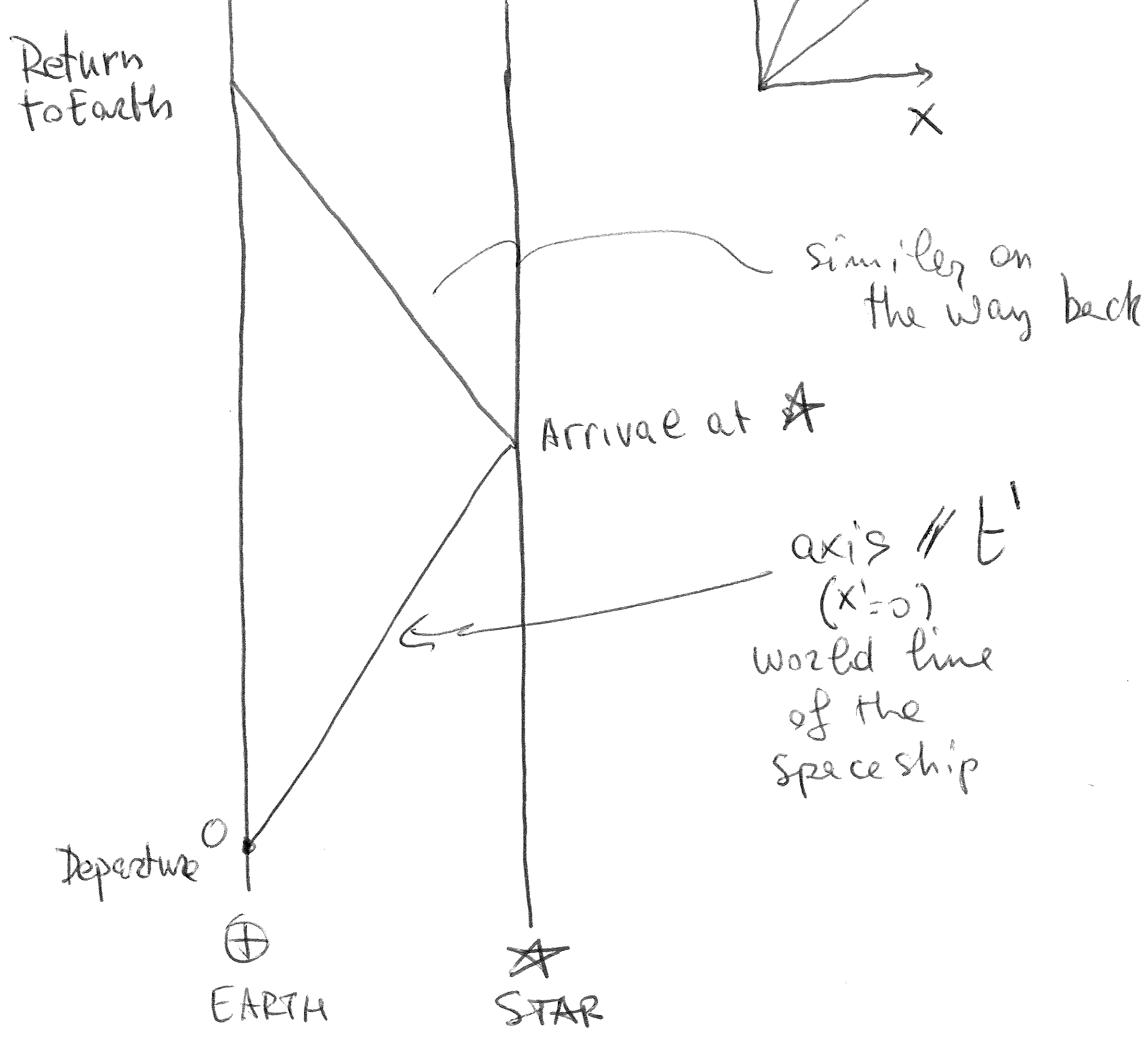
b) $d = vT = 0.95 \times \underbrace{\left(30 \frac{cm}{ns}\right)}_{= c} \times 83.3\,ns$

$= 23.7\ m$

3.6

From Earth frame:

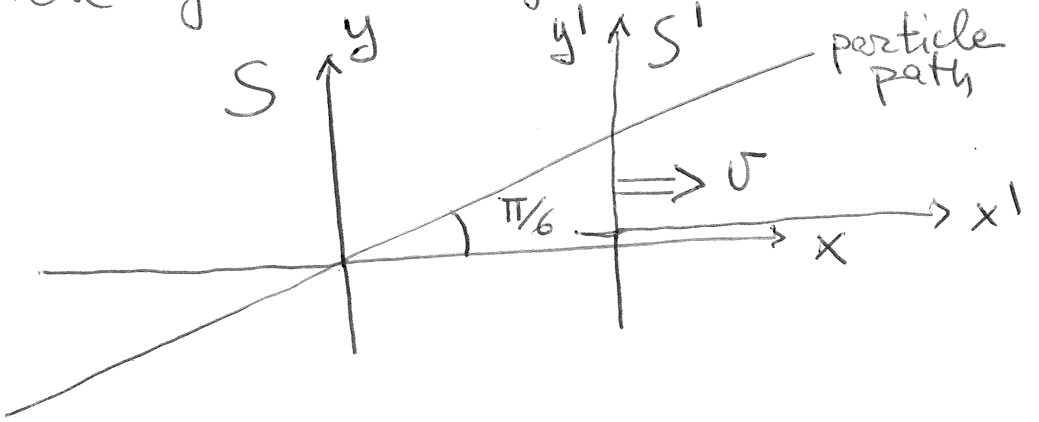




See Twin Paradox on Wikipedia.

4.2

Assume for simplicity that the particle goes through the origin of S'



in S: $v_x = u \times \cos \frac{\pi}{6} = \frac{c}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} \cdot c$

$$\Rightarrow x = v_x t = \frac{\sqrt{3}}{4} \cdot ct$$

transforming to S':

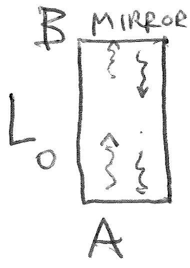
$$x' = \gamma(x - vt) = \gamma\left(\frac{\sqrt{3}}{4}c - v\right)t$$

But if we want the trajectory to be $\perp$ to the x' axis, it must be $x' = 0$ all the time.

$$\Rightarrow v = \frac{\sqrt{3}}{4} \cdot c$$

4.3

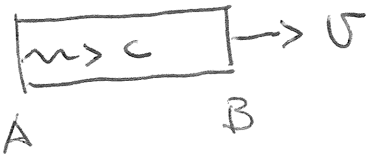
Suppose the clock at rest has length L_0:



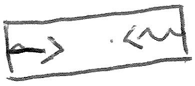
Time between a full trip $A \to B \to A$

$$T_0 = \frac{2L_0}{c}$$

(a) $L = L_0/\gamma$



$$\text{time } A \to B = \frac{L}{c-v}$$

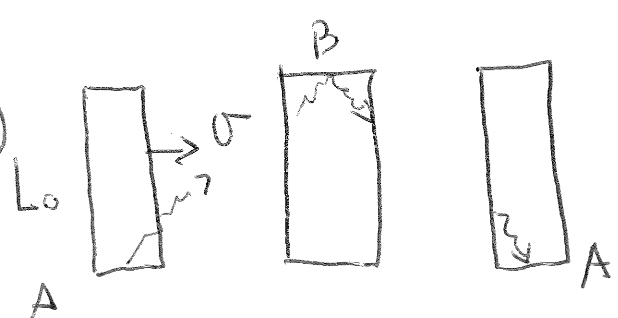


$$\text{time } B \to A = \frac{L}{c+v}$$

$$\text{Total time } T = L\left(\frac{1}{c-v} + \frac{1}{c+v}\right) =$$

$$= \frac{L_0}{\gamma} \cdot \frac{2c}{c^2 - v^2} = \frac{2L_0}{c} \cdot \gamma = T_0 \gamma$$

(b)

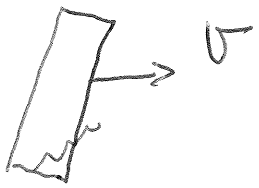


$$\text{Time } A \to B \to A = 2 \times \text{time } A \to B =$$

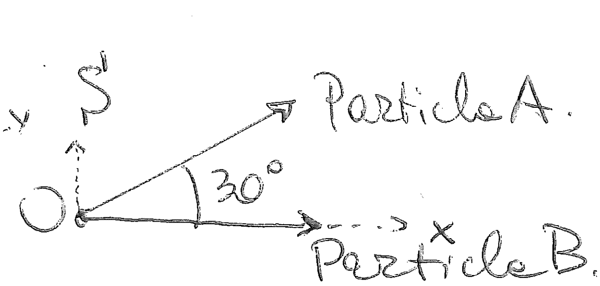
$$= \frac{2L_0}{\sqrt{c^2 - v^2}} = \frac{2L_0}{c} \frac{1}{\sqrt{1 - v^2/c^2}} = T_0 \gamma$$

They are both the same and are ~~the~~ those predicted.

(c) The same MOST work for any inclination.



4.4



The velocity of particle A in S, chosing O as the origin is $v_x = \frac{\sqrt{3}}{2} v$, $v_y = \frac{1}{2} v$

To get the relative velocity we boost to the rest frame of Particle B.

$$v_x' = \frac{v_x - v}{1 - v \cdot v_x} \qquad v_y' = \frac{v_y}{\gamma(v)(1 - v v_x)} \Rightarrow$$

$$v_x' = \frac{\left(\frac{\sqrt{3}}{2} - 1\right) v}{1 - \frac{\sqrt{3}}{2} v^2} \qquad v_y' = \frac{\frac{1}{2} v}{\gamma(v)\left(1 - \frac{\sqrt{3}}{2} v^2\right)}$$

$$v' = \sqrt{v_x'^2 + v_y'^2} = \frac{1}{1 - \frac{\sqrt{3}}{2} v^2} \cdot \sqrt{\left(\frac{\sqrt{3}}{2} - 1\right)^2 v^2 + \frac{1}{4} v^2 (1 - v^2)}$$

$$= \frac{\sqrt{(8 - 4\sqrt{3}) v^2 - v^4}}{2 - \sqrt{3} v^2} = v \cdot \frac{\sqrt{(8 - 4\sqrt{3}) - v^2}}{2 - \sqrt{3} v^2}$$

4.6

$x = \frac{k}{3} t^3 \Rightarrow v = k t^2 \Rightarrow a = 2kt$

This can only be valid for

$|v| < c \Rightarrow |t| < \sqrt{c/|k|}$

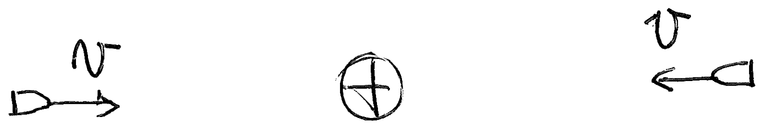
In this time interval:

$$\alpha = \gamma^3 a = \frac{2kt}{\left(1 - \left(\frac{kt^2}{c}\right)^2\right)^{3/2}}.$$

Note that also α does not make sense for $|t| > \sqrt{c/|k|}$.

4.8

From Earth

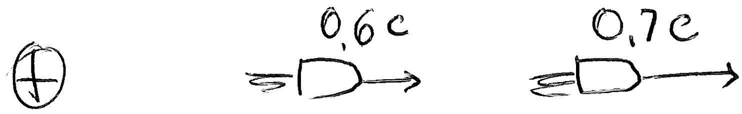


We know $\dfrac{v+v}{1+v^2/c^2} = 0.9\,c$

$\Rightarrow v = 0.63\,c$

4.9

Earth Frame



Boost to

$$v = \frac{0{,}7c - 0.6c}{1 - 0{,}7 \cdot 0.6 \frac{\cancel{c}^2}{\cancel{c}^2}} = 0{,}17c$$

5.1

The full relativistic formula is:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Expand to II order in v:

$$u = u' + \left(1 - \frac{u'^2}{c^2}\right)v - \frac{u'}{c^2}\left(1 - \frac{u'^2}{c^2}\right)v^2$$

$$= u' + kv - u'k\frac{v^2}{c^2}$$

The relative correction is

$$\frac{u'k\, v^2/c^2}{u' + kv} = \frac{\frac{c}{n}\left(1 - \frac{1}{n^2}\right)\cdot\frac{v^2}{c^2}}{\frac{c}{n} + \left(1 - \frac{1}{n^2}\right)v} =$$

$$= \frac{\frac{1}{n}\left(1 - \frac{1}{n^2}\right)\frac{v^2}{c^2}}{\frac{1}{n} + \left(1 - \frac{1}{n^2}\right)\frac{v}{c}} \underset{(v \ll c)}{\simeq} \left(1 - \frac{1}{n^2}\right)\frac{v^2}{c^2} \simeq 5 \times 10^{-16}\ \%$$

5.2 In this (very unrealistic!)

(a) case, we must use the exact formula:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\frac{c}{n} + \frac{c}{2}}{1 + \frac{c/n \cdot c/2}{c^2}} =$$

$$= \frac{\frac{1}{n} + \frac{1}{2}}{1 + \frac{1}{2n}} c = \frac{\frac{3}{4} + \frac{1}{2}}{1 + \frac{3}{8}} c = \frac{10}{11} c$$

(b) Fizeau's formula:

$$u = u' + \left(1 - \frac{1}{n^2}\right) v =$$

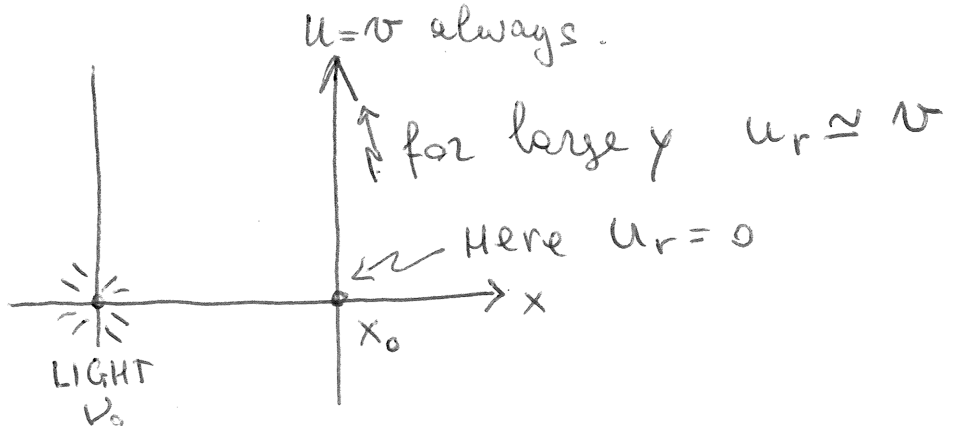
$$= \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)\frac{c}{2} = \left(\frac{3}{4} + \frac{1}{2}\left(1 - \frac{9}{16}\right)\right)c$$

$$= \frac{31}{32} \cdot c$$

5.4

We use the same formula we found in problem 5.3:

$$\nu = \gamma(u)(1-u_r)\,\nu_0$$



So, near the x axis:

$$\nu = \gamma(v)\,\nu_0 = \frac{\nu_0}{\sqrt{1-v^2}} \qquad (c=1 \text{ units}).$$

Far from the x axis:

$$\nu = \gamma(v)(1-v)\,\nu_0 = \sqrt{\frac{1-v}{1+v}}\,\nu_0 .$$

5.6

$$\nu = \frac{c}{\lambda} \qquad\qquad \nu_0 = \frac{c}{\lambda_0}$$

$\oplus \leftsquigarrow$ $\qquad\qquad$ $\star \longrightarrow v$

(a) Doppler formula:

$$\frac{c}{\lambda_0} = \gamma(v)\left(1+\frac{v}{c}\right)\frac{c}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}}\,\frac{c}{\lambda}$$

$$\Rightarrow \sqrt{\frac{1+v/c}{1-v/c}} = \frac{\lambda}{\lambda_0} \Rightarrow v = \frac{\left(\frac{\lambda}{\lambda_0}\right)^2 - 1}{\left(\frac{\lambda}{\lambda_0}\right)^2 + 1}\,c = 0.969\,c$$

very close to c!

(b) $$d \simeq \frac{c}{H} = \frac{3\times 10^8\ m/s}{72\times 10^3\ \frac{m}{s\cdot Mpc}} = 4.2\times 10^3\ Mpc$$

(Note: the size of the "observable universe is $\sim 15\times 10^3$ Mpc).

5.6 (c=1 units)

Exact formula $\frac{\nu_0}{\nu} = \sqrt{\frac{1+u}{1-u}} \cong (\text{Taylor})\ 1 + u + \frac{1}{2}u^2$

Non relat. formula: $\frac{\nu_0}{\nu} = 1 + u$.

$$\text{Error} = \frac{\frac{1}{2}u^2}{1+u} \simeq \frac{1}{2}u^2 = \frac{1}{2} \cdot 0.01^2 = 0.005\,\%_0$$

5.7

ν

$\oplus$

$\nu_0 = 3\nu$

$\bigstar \longrightarrow v$

$$\frac{\nu_0}{\nu} = 3 = \sqrt{\frac{1+v/c}{1-v/c}}$$

$$\Rightarrow v = 0.8\,c$$

6.1

a) OK

b) NO the η is covariant the dx are contravariant, so η must transform with Λ^{-1}

c) NO One should write $A^{\mu}{}_{\mu}$

d) NO ds^2 is a differential, not a number! (And it also has units...)

e) NO It can be valid in ONE FRAME but cannot be valid in all.

f) NO It should be $\eta_{\mu\nu}\eta^{\mu\rho} = \delta^{\rho}_{\nu}$

g) OK

h) NO : μ is summed on the L.H.S and free on the R.H.S.

6.3

a) In $\epsilon^{\alpha\beta\gamma\delta}$, the indices $\alpha, \beta, \gamma, \delta$ can take values in $\{0,1,2,3\}$.

If there is one repeated index (ex. index 0) then the component vanishes: $\epsilon^{0012} = -\epsilon^{0012}$

$\Rightarrow \epsilon^{0012} = 0$.

If all indices are different, I can use asymmetry to relate them to $\epsilon^{0123} = 1$. Eg. $\epsilon^{2103} = -\epsilon^{2013} =$

$= +\epsilon^{0213} = -\epsilon^{0123} = -1$ and so on..

b) $\epsilon'^{\alpha\beta\gamma\delta} = \Lambda^{\alpha}_{\ \mu} \Lambda^{\beta}_{\ \nu} \Lambda^{\gamma}_{\ \rho} \Lambda^{\delta}_{\ \sigma} \epsilon^{\mu\nu\rho\sigma}$

$= \cancel{\det \Lambda}\, \epsilon^{\alpha\beta\gamma\delta}$

$= \epsilon^{\alpha\beta\gamma\delta}$.

6.4 There are of course many possible solutions. Some examples:

a) $$\begin{pmatrix}1\\2\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}$$

b) $$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}$$

c) $$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}2\\1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\1\\0\\0\end{pmatrix}$$

d) $$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

e) $$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 0$$

<u>6.5</u>

a) $V_1 \cdot V_1 = 1^2 - 2^2 - (-2)^2 - 0^2 = -7 < 0$
spacelike. ("Present").

b) $V_2 \cdot V_2 = (-2)^2 - 1^2 - (-2)^2 = -1 < 0$
Space like

c) $V_3 \cdot V_3 = 1^2 - (-2)^2 - 1^2 = -4 < 0$
also space like.

d) $V_1 + V_3 = (2, 0, -1, 0)$
Time like <u>future</u> $(2 > 0)$

e) $V_2 - V_1 = (-3, -1, 0, 0)$
time like <u>past</u> $(-3 < 0)$

f) $V_1 + V_2 + V_3 = (0, 5, -3, 0)$
space like.

6.6

$c = 1$ units.

(a) We know: $U^\mu U_\mu = 1$ [1]

$A^\mu U_\mu = 0$ [2]

$A^\mu A_\mu = -\alpha^2$ [3]

In addition, we are given: $\frac{dA^\mu}{d\tau} = \phi U^\mu$.

$\Rightarrow U^\mu = \frac{1}{\phi} \frac{dA^\mu}{d\tau}$. Inserting in [1]:

$$\frac{1}{\phi} \frac{dA^\mu}{d\tau} U_\mu = 0 \quad \Rightarrow \quad \frac{dA^\mu}{d\tau} U_\mu = \phi.$$

taking $\frac{d}{d\tau}$ [2] $\Rightarrow$ $\frac{dA^\mu}{d\tau} U_\mu + A^\mu \frac{dU_\mu}{d\tau} = 0$

$$\Rightarrow \phi + A^\mu A_\mu = 0 \quad \Rightarrow \quad \phi - \alpha^2 = 0$$

$$\Rightarrow \alpha = \sqrt{\phi}.$$

$\underline{6.8}$

$$\begin{cases} x = C_x t \\ y = C_y \sqrt{t} \\ z = 0 \end{cases} \Rightarrow \begin{cases} u_x = C_x \\ u_y = C_y \frac{1}{2\sqrt{t}} \\ u_z = 0 \end{cases} \Rightarrow \begin{cases} a_x = 0 \\ a_y = -\frac{1}{4} C_y t^{-\frac{3}{2}} \\ a_z = 0 \end{cases}$$

$c = 1$ UNITS .

$$\gamma = \frac{1}{\sqrt{1 - C_x^2 - C_y^2/4t}}$$

$$U^\mu = \frac{1}{\sqrt{1 - C_x^2 - C_y^2/4t}} \left(1, C_x, C_y \frac{1}{2\sqrt{t}}, 0\right)$$

$$\dot{\gamma} = \frac{-C_y^2/8t^2}{\left(1 - C_x^2 - C_y^2/4t\right)^{\frac{3}{2}}}$$

$$A^\mu = \frac{1}{\sqrt{1 - C_x^2 - C_y^2/4t}} \left(\frac{-C_y^2/8t^2}{\left(1 - C_x^2 - \frac{C_y^2}{4t}\right)^{\frac{3}{2}}}, \frac{-C_x C_y^2/8t^2}{\left(1 - C_x^2 - \frac{C_y^2}{4t}\right)^{\frac{3}{2}}}, \right.$$

$$\left. \frac{-\frac{C_y^3}{16} t^{-\frac{5}{2}}}{\left(1 - C_x^2 - \frac{C_y^2}{4t}\right)^{\frac{3}{2}}} + \frac{-\frac{1}{4} C_y t^{-\frac{3}{2}}}{\left(1 - C_x^2 - \frac{C_y^2}{4t}\right)^{\frac{1}{2}}}, 0 \right) =$$

$$= \frac{1}{\left(1 - C_x^2 - \frac{C_y^2}{4t}\right)^2} \left(-\frac{C_y^2}{8t^2}, -\frac{C_x C_y^2}{8t^2}, -\frac{1}{4} C_y \left(1 - C_x^2\right) \frac{1}{t^{3/2}}, 0 \right)$$

$$U^\mu U_\mu = \frac{1}{1-c_x^2-\frac{c_y^2}{4t}}\left(1-c_x^2-\frac{c_y^2}{4t}\right) = 1.$$

$$A^\mu U_\mu = \frac{1}{\left(1-c_x^2-\frac{c_y^2}{4t}\right)^{\frac{5}{2}}}\left(-\frac{c_y^2}{8t^2}+\frac{c_x^2 c_y^2}{8t^2}+\right.$$

$$\left.+\frac{1}{8}c_y^2\left(1-c_x^2\right)\cdot\frac{1}{t^2}\right) = 0.$$

6.9

$$E = h\nu = \hbar\omega$$
$$\mathbb{P} = h\mathbb{k} = \hbar\mathbb{K}$$

$$E^2 - \mathbb{P}^2 = m^2 = \hbar^2(\omega^2 - \mathbb{K}^2)$$

$$\Rightarrow \omega^2 - \mathbb{K}^2 = \frac{m^2}{\hbar^2} = \frac{4\pi^2}{\ell^2}$$

$$\frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\left(\sqrt{4\pi^2/\ell^2 + k^2}\right) =$$

$$= \frac{1}{2}\frac{2k}{\left(\frac{4\pi^2}{\ell^2} + k^2\right)^{\frac{1}{2}}} = \frac{k}{\omega} = \frac{P}{E} = u.$$

Also as a vector:

$$\nabla_{\mathbb{K}}\omega = \frac{\mathbb{K}}{\omega} = \frac{\mathbb{P}}{E} = \mathbb{u}.$$

7.10 This is a THREE BODY DECAY (NOT a TWO BODY DECAY like the previous exercises).

This means that the final energies are NOT fixed but depend on the relative angles.

The max energy for one of the three particles is attained when the other two move in the opposite direction as a single particle with mass $m_1 + m_2$.

This is a rather intuitive and easy to remember result but it is a bit tricky to prove rigorously (try it!).

Assuming that, the problem is reduced to a 2 body decay:

Case 1: $K^- \to e^- + \underbrace{\pi^0 + \bar{\nu}_e}_{m = m_\pi}$:

$$P_{e^-, max} = \frac{m_K^2 - m^2}{2 m_K} = 231. \text{ MeV}$$

$$E_{e^-, max} = \sqrt{m_e^2 + P_{e^- max}^2} \simeq P_{e^- max} = 231. \text{ MeV}.$$

Case 2: $\mu^- \to e^- + \underbrace{\nu_\mu + \bar{\nu}_e}_{m = 0}$

$$P_{e^- max} = \frac{1}{2} m_\mu = 52.5 \text{ MeV}$$

$$E_{e^- max} \cong P_{e^- max} = 52.5 \text{ MeV}.$$

7.11

$\boxed{c=1}$

$$\underbrace{p^\mu_{\pi^-} + p^\mu_p}_{\text{square in the lab frame}} = \underbrace{p^\mu_{K^0} + p^\mu_{\Sigma^0}}_{\text{square in the CM frame}}$$

square in the lab frame = square in the CM frame:

since they are Lorentz invariant.

$$(E_{\pi^-} + m_p)^2 - p^2 = (m_{K^0} + m_{\Sigma^0})^2$$

$$\underbrace{E_{\pi^-}^2 + 2m_p E_{\pi^-} + m_p^2 - p^2}_{E_{\pi^-}^2 - p^2 \,=\, m_{\pi^-}^2} = (m_{K^0} + m_{\Sigma^0})^2$$

$$2m_p E_{\pi^-} + m_p^2 + m_{\pi^-}^2 = (m_{K^0} + m_{\Sigma^0})^2$$

$$\Rightarrow E_{\pi^-} = \frac{(m_{K^0} + m_{\Sigma^0})^2 - m_p^2 - m_{\pi^-}^2}{2m_p}$$

$$= 1045 \text{ MeV}.$$

7, 13

C=1

Case 1: $pp \to ppp\bar{p}$:

$$E_{threshold} = \frac{(4m_p)^2 - m_p^2 - m_p^2}{2m_p} = 7m_p$$

$(\simeq 6{,}6\,\text{GeV})$

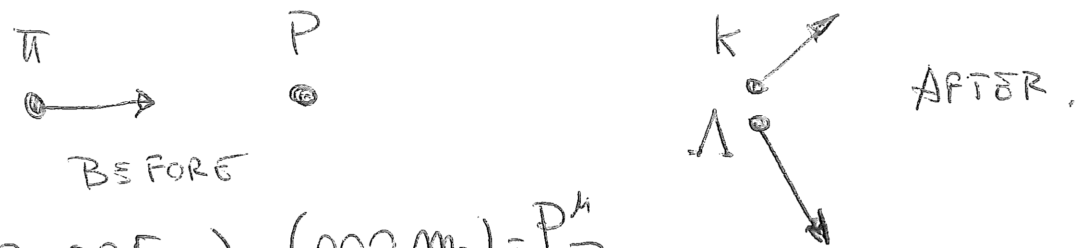
Case 2: $e^+e^- \to p\bar{p}$

$$E_{threshold} = \frac{(2m_p)^2 - m_e^2 - m_e^2}{2m_e} = \frac{2m_p^2 - m_e^2}{2m_e}$$

$(\simeq 1.7\ \text{TeV})$

7.15

$$\boxed{\pi + P \rightarrow K + \Lambda}$$



$$\underbrace{P^\mu_\pi = (P_\pi, 0, 0, E_\pi) \quad (0\,0\,0\,m_p) = P^\mu_p}_{\text{in the LAB frame}}$$

The MINIMUM energy of K and Λ in the CM frame is their rest mass:

$$\underbrace{P^\mu_K = (0\,0\,0\,m_K) \quad P^\mu_\Lambda = (0\,0\,0\,m_\Lambda)}_{\text{IN the CM frame}}.$$

$$\underbrace{(P^\mu_\pi + P^\mu_p)^2 = (P^\mu_K + P^\mu_\Lambda)^2}_{\text{Lorentz invariant, valid in ANY FRAME}}.$$

$$(E_\pi + m_p)^2 - P_\pi^2 = (m_K + m_\Lambda)^2$$

$$m_p^2 + E_\pi^2 + 2m_p E_\pi - P_\pi^2 = (m_K + m_\Lambda)^2$$

$$m_p^2 + 2m_p E_\pi + m_\pi^2 = (m_K + m_\Lambda)^2$$

$$E_\pi = \frac{(m_K + m_\Lambda)^2 - m_\pi^2 - m_p^2}{2m_p} = 909 \text{ MeV}.$$

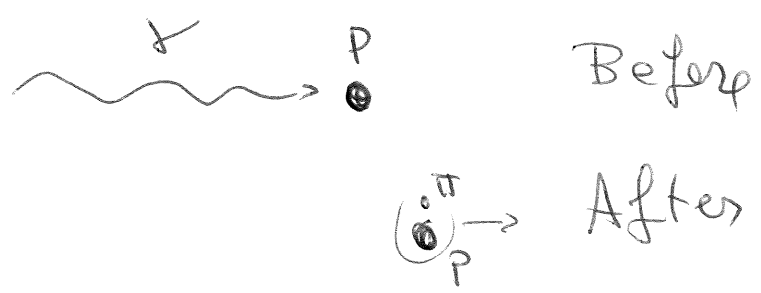
7.17 $\boxed{c=1}$ $\gamma p \to \pi^0 p$

$$E_{threshold} = \frac{(m_{\pi^0} + m_p)^2 - 0 - m_p^2}{2 m_p} =$$

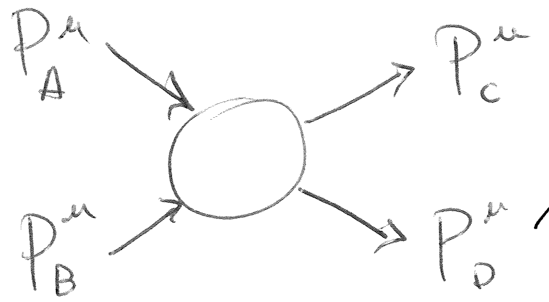
(a)

$$= \frac{m_{\pi^0}^2 + 2 m_p m_{\pi^0}}{2 m_p} \simeq 145 \text{ MeV}$$

(b) Just above threshold the outgoing particles have almost no relative velocity, i.e. they move almost together as a particle of mass $m_{\pi^0} + m_p$ and momentum = incoming photon momentum:



7.18



Reversed sign compared to compendium ($P_{C,D}^\mu \to -P_{C,D}^\mu$)

$$P_A^\mu + P_B^\mu = P_C^\mu + P_D^\mu$$

$s = (P_A + P_B)^2$ is the Energy^2 in the center of mass AND it is lorentz invariant $\Rightarrow$ $s > 0$ in all frames.

Now: $s + t + u = P_A^2 + 2P_A P_B + P_B^2 +$

$$+ P_A^2 - 2P_A P_C + P_C^2 +$$

$$+ P_B^2 - 2P_B P_C + P_C^2 =$$

$$= 2m_A^2 + 2m_B^2 + 2m_C^2 + 2P_A P_B - 2P_A P_C - 2P_B P_C \quad [1]$$

But also, writing the conservation of momentum as: $P_A^\mu + P_B^\mu - P_C^\mu = P_D^\mu$ and doing the lorentz square:

$$P_A^2 + P_B^2 + P_C^2 + 2P_A \cdot P_B - 2P_A \cdot P_C - 2P_B \cdot P_C = P_D^2$$

$$\Rightarrow 2P_A \cdot P_B - 2P_A \cdot P_C - 2P_B \cdot P_C = m_D^2 - m_A^2 - m_B^2 - m_C^2 \quad [2]$$

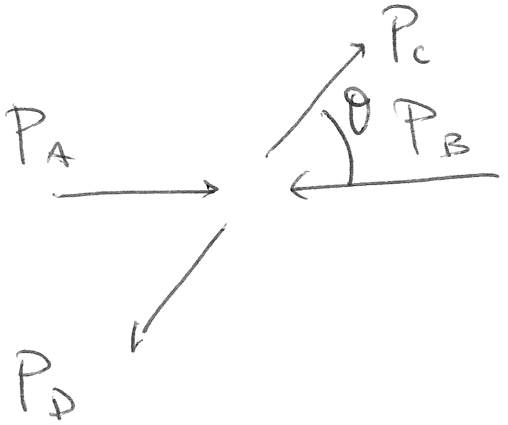
Substituting [2] into [1] we get:

$$s+t+u = m_A^2 + m_B^2 + m_C^2 + m_B^2 .$$

Let now all masses be equal to m and compute t in the CM frame (it is a Lorentz invariant).

Chose $P_A^\mu = (E, P, 0, 0)$ by rotation

$$\Rightarrow P_C^\mu = (E, p\cos\theta, p\sin\theta, 0)$$



$$t = (P_A - P_C)^2 = (0, \underbrace{p(1-\cos\theta), -p\sin\theta, 0}_{\text{SPACE LIKE!}})^2 =$$

$$= -p^2(1-\cos\theta)^2 - (-p\sin\theta)^2 =$$

$$= -2p^2(1-\cos\theta) \leq 0$$

Same for u by exchange $C \leftrightarrow D$.

7.19

We can write $v = 0.4\,c$ and set $c = 1$ everywhere.

The rocket equation for a photon exhaust is then:

$$\frac{M_{before}}{M_{after}} = \left(\frac{1+v}{1-v}\right)^{\frac{1}{2}} = 1.53$$

7.1

For each proton:

$$T = mc^2\gamma - mc^2 = mc^2\left(\frac{1}{\sqrt{1-0{,}6^2}} - 1\right)$$

$$= 0{,}25\, mc^2$$

Total energy $= 2.5 \times 10^8\, mc^2 =$

$= 2.5 \times 10^8 \times 938\ \text{MeV} =$

$= 2{,}3 \times 10^{11}\ \text{MeV} = 3.7 \times 10^{-2}\ \text{J}.$

7.20

Same equation as problem 7.19:

$$\frac{M_{before}}{M_{after}} = \sqrt{\frac{1+v}{1-v}}$$

Now we want $\frac{M_{before}}{M_{after}} = 2$

$$\Rightarrow \quad v = \frac{3}{5} \ (\times c) = 1{,}8 \times 10^{8} \ m/s.$$

7,2

$$T = (\gamma - 1) mc^2 = 10 mc^2$$

$$\Rightarrow \gamma = 11 \Rightarrow v = \sqrt{\frac{121}{122}}\, c \cong 0{,}996\, c$$

7.4 $\nabla\psi = i\frac{\mathbb{P}}{\hbar}\psi$

$$\nabla\cdot\nabla\psi = \frac{(i)^2}{\hbar^2}\mathbb{P}\cdot\mathbb{P}\psi = -\frac{p^2}{\hbar^2}\psi$$

$$\dot{\psi} = -i\frac{E}{\hbar}\psi,$$

(a)

Schrödinger:

$$-\frac{\hbar^2}{2m}\left(-\frac{p^2}{\hbar^2}\psi\right) = i\hbar\left(-i\frac{E}{\hbar}\psi\right)$$

$$\Rightarrow \frac{p^2}{2m}\psi = E\psi \quad \checkmark$$

(b) Klein-Gordon:

$$-\hbar^2c^2\left(-\frac{p^2}{\hbar^2}\psi\right) + m^2c^4\psi = -\hbar^2\left(-i\frac{E}{\hbar}\right)^2\psi$$

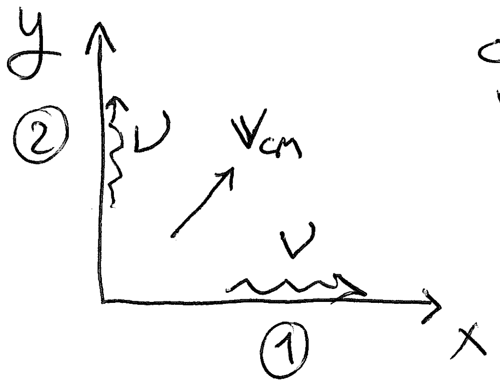
$$c^2p^2\psi + m^2c^4\psi = +E^2\psi \quad \checkmark$$

(c) Set $c=\hbar=1$ units and rewrite

$$\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + m^2\psi = 0$$

$$\Rightarrow \partial_\mu\partial^\mu\psi + m^2\psi = 0 \quad \checkmark$$

7.7



$c = 1$ units.

$$P^{\mu}_{TOT} = P^{\mu}_{1} + P^{\mu}_{2} = (h\nu, h\nu, 0, 0) + (h\nu, 0, h\nu, 0) = (2h\nu, h\nu, h\nu, 0)$$

$$\mathbb{V}_{CM} = \frac{\mathbb{P}_{TOT}}{E_{TOT}} = \frac{(h\nu, h\nu, 0)}{2h\nu} = \frac{1}{2}(1,1,0)$$

Note that $\mathbb{V}_{CM} \cdot \mathbb{V}_{CM} = \frac{1}{4}(1+1) = \frac{1}{2} < 1$

$$\Rightarrow E_p = \sqrt{m_p^2 + p^2} = 945 \text{ MeV}$$

$$E_\pi = \sqrt{m_\pi^2 + p^2} = 182 \text{ MeV}$$

$$v_p = \frac{p}{E_p} = 0.124\, c$$

$$v_\pi = -\frac{p}{E_\pi} = -0.641\, c$$

7.8

$c=1$

$$\pi \longleftarrow \overset{\Lambda}{\bullet} \longrightarrow p \qquad \longrightarrow x$$

$$p_\Lambda^\mu = (m_\Lambda, 0, 0, 0)$$
$$p_\pi^\mu = (E_\pi, p_\pi, 0, 0)$$
$$p_p^\mu = (E_p, p_p, 0, 0)$$

Choose the x axis. Drop the other two

$$p_\Lambda^\mu = p_\pi^\mu + p_p^\mu \Rightarrow \begin{cases} m_\Lambda = E_\pi + E_p \\ 0 = p_\pi + p_p \end{cases}$$

$\Rightarrow$ set $p_p = -p_\pi = p$

$$m_\Lambda = \sqrt{m_\pi^2 + p^2} + \sqrt{m_p^2 + p^2}$$

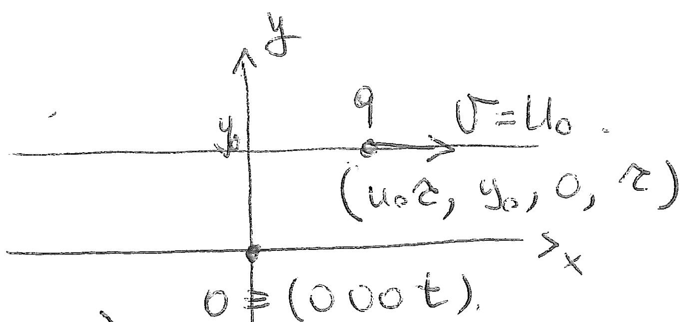
Solve for p: (move one $\sqrt{\ }$ to the left & square. Then square again).

$$p = \frac{\sqrt{m_\Lambda^4 + m_\pi^4 + m_p^4 - 2m_\Lambda^2 m_\pi^2 - 2m_\Lambda^2 m_p^2 - 2m_\pi^2 m_p^2}}{2m_\Lambda}$$

$= 117\ \text{MeV}$

Same as in the Compendium

8.10



$$U^\mu = \gamma(u_0)(u_0, 0, 0, 1)$$

$$R^\mu = (-u_0\tau, -y_0, 0, t-\tau)$$

where τ is the time at which the charge affects the origin:

$$R_\mu R^\mu = (t-\tau)^2 - u_0^2 - u_0^2\tau^2 = 0$$

$$\Rightarrow \tau = \frac{t - \sqrt{t^2 - (1-u_0^2)(t^2 - y_0^2)}}{1-u_0^2}$$

$$\phi^4 = q\,\frac{\gamma}{\gamma(t-\tau) + \gamma u_0^2 \tau} = \frac{q}{t - \tau + u_0^2\tau}$$

where τ is given by the eq. above.

8.1

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{bmatrix}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{bmatrix}$$

$$F_{\mu\nu}F^{\mu\nu} = F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + \\ + F_{12}F^{12} + F_{13}F^{13} + F_{23}F^{23} + \\ \underbrace{F_{10}F^{10} + \dots F_{32}F^{32}}_{\text{same as above}} =$$

since $F_{10}F^{10} = (-F_{01})(-F^{01})$

$$= 2(F_{01}F^{01} + \dots F_{23}F^{23}) = 2(-E_x^2 - E_y^2 \\ -E_z^2 + B_z^2 + B_y^2 + B_x^2) = -2(\mathbb{E}\cdot\mathbb{E} - \mathbb{B}\cdot\mathbb{B})$$

8.2 Let us write $\mathbb{F} = q(\mathbb{E} + \mathbb{v}\times\mathbb{B})$.

Recall that $\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d}{dt}(1-v^2)^{-\frac{1}{2}} =$

$= -\frac{1}{2}(1-v^2)^{-\frac{3}{2}}\cdot(-2v\dot{v}) = \gamma^3 v\dot{v} = \gamma^3 \mathbb{v}\dot{\mathbb{v}}$

Also recall that m is a constant.

$$\frac{d}{dt}(\gamma m \mathbb{v}) = \dot{\gamma} m \mathbb{v} + \gamma m \dot{\mathbb{v}} = \mathbb{F}$$
$$= \gamma^3 m(\mathbb{v}\cdot\dot{\mathbb{v}})\mathbb{v} + \gamma m \dot{\mathbb{v}} = \mathbb{F} \quad [1]$$

Take the $\cdot$ product of [1] with $\mathbb{v}$:

$$\gamma^3 m\, \mathbb{v}\cdot\dot{\mathbb{v}}\, v^2 + \gamma m\, \mathbb{v}\cdot\dot{\mathbb{v}} = \mathbb{v}\cdot\mathbb{F}$$

$$\gamma m\, \mathbb{v}\cdot\dot{\mathbb{v}}\,(\underbrace{\gamma^2 v^2 + 1}_{=\gamma^2}) = \mathbb{v}\cdot\mathbb{F} = q\,\mathbb{v}\cdot\mathbb{E}$$

(since $\mathbb{v}\cdot\mathbb{v}\times\mathbb{B} = 0$)

$$\Rightarrow \gamma^3 m\, \mathbb{v}\cdot\dot{\mathbb{v}} = q\,\mathbb{v}\cdot\mathbb{E}$$

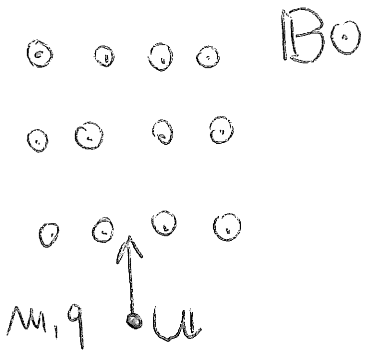
Substituting back in [1]:

$$q(\mathbb{v}\cdot\mathbb{E})\mathbb{v} + \gamma m \dot{\mathbb{v}} = \mathbb{F}$$

$$\Rightarrow \gamma m \dot{\mathbb{v}} = \mathbb{F} - q(\mathbb{v}\cdot\mathbb{E})\mathbb{v} \quad \checkmark$$

$$\gamma \frac{dm\mathbb{v}}{dt} = q(\mathbb{E} + \mathbb{v}\times\mathbb{B} - (\mathbb{v}\cdot\mathbb{E})\mathbb{v}).$$

8.3



$$\mathbb{f} = q\,\mathbb{u} \times \mathbb{B}$$

(a)

Relativistic 3-mom. and 3-force.

Once again: $F^\mu = \dfrac{dP^\mu}{d\tau} = \gamma\left(\dfrac{dE}{dt}, \dfrac{d\mathbb{P}}{dt}\right) = \gamma\left(\dfrac{dE}{dt}, \mathbb{f}\right)$

$$F^\mu = \frac{dP^\mu}{d\tau} = m\frac{dU^\mu}{d\tau} + \cancel{\frac{dm}{d\tau}}U^\mu$$

$\cancel{\frac{dm}{d\tau}} = 0$ — PRESERVING The rest mass!

$$\Rightarrow F^\mu U_\mu = 0 = \frac{dE}{dt} - \mathbb{u}\cdot\mathbb{f}$$

For this case $\mathbb{u}\cdot\mathbb{f} = 0 \Rightarrow E$ conserved.

also $\mathbb{f} = \dfrac{d\mathbb{P}}{dt} = \dfrac{d}{dt}(E\,\mathbb{u}) = E\cdot\dfrac{d\mathbb{u}}{dt} = E\,\mathbb{a}$.

thus $\mathbb{a}\cdot\mathbb{u} \propto \mathbb{f}\cdot\mathbb{u} = 0 \Rightarrow u$ constant.

$$\left(\frac{du^2}{dt} = 2u\dot{u} \equiv 2\,\mathbb{u}\cdot\mathbb{a} = 0\right)$$

Now the eq. of motion is reduced to the same eq. we have non rel. but with the mass replaced by the (constant) Energy E/c^2)

$\Rightarrow \quad q\, \mathbb{u} \times \mathbb{B} = E\, \mathbb{a}. \qquad (\mathbb{B} = B\, e_z)$

Take the ansatz $\mathbb{u} = \underset{\text{const.}}{u}(\sin\theta\, e_x + \cos\theta\, e_y)$

$\Rightarrow \mathbb{a} = u\dot{\theta}(\cos\theta\, e_x - \sin\theta\, e_y)$

$\Rightarrow q u B(-\sin\theta\, e_y + \cos\theta\, e_x) = E u \dot{\theta}(\cos\theta\, e_x - \sin\theta\, e_y)$

$\Rightarrow \dot{\theta} = \frac{qB}{E} = \omega$ also constant, $T = \frac{2\pi E}{qB}$

integrating $\mathbb{u} = u(\sin\omega t\, e_x + \cos\omega t\, e_y)$

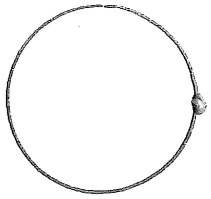
gives $\mathbb{r} = \frac{u}{\omega}(-\cos\omega t\, e_x + \sin\omega t\, e_y)$

$\Rightarrow$ Radius $R = \frac{u}{\omega} = \frac{uE}{qB}$ also const.

Putting units back and writing $m\gamma c^2$ instead of E:

$$R = \frac{c\, m\, u\, \gamma(u)}{qB}, \quad T = \frac{2\pi c\, m\, \gamma(u)}{qB}$$

[Note that the formula for $\gamma = 1$ can be obtained by non rel. considerations



$$m\frac{v^{\cancel{2}}}{R} = qB\cancel{v} \Rightarrow \omega = \frac{qB}{m}.$$]

If the particle has a u_z component such component remains const. since $(\mathbf{u} \times \mathbf{B})_z = 0$ still.

$\Rightarrow$ helix w/ same radius.

(b) From $R = \dfrac{c\, m\, u\, \gamma}{qB}$ we get $R = \dfrac{cP}{qB} \Rightarrow cP = q\, RB$

Remember that $1\,\text{Tesla} = \dfrac{1\,\text{kg}}{1\,\text{Coulomb} \times s}$

So to get the units right I have to multiply the RHS. by an extra "c":

$$cP = qcRB = 1.6\times10^{-19}\,C \times 3\times10^{8}\,\frac{m}{s} \times \frac{R}{\text{meter}} \times m \times \frac{B}{\text{Tesla}} \times T$$

$$= 4.8\times10^{-11}\,C \cdot \frac{m^2 T}{s}\, \frac{R}{\text{meter}} \times \frac{B}{\text{Tesla}} =$$

$$= 4.8\times10^{-11}\,J\, \frac{R}{\text{meter}} \cdot \frac{B}{\text{Tesla}} = 300\,\text{MeV}\, \frac{R}{\text{meter}}\, \frac{B}{\text{Tesla}},$$

8.6 ρ_0 is scalar and U^μ is a 4-vector

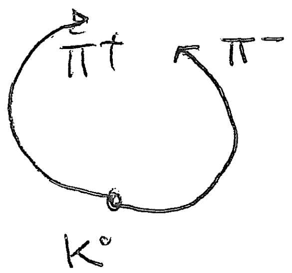
$\Rightarrow \rho_0 U^\mu$ is a 4 vector.

8.7 Using the formula in problem 8.3.

$$\frac{cp}{\text{MeV}} = 300 \times \frac{B}{\text{Tesla}} \times \frac{R}{\text{meter}}$$

$$\Rightarrow cp = 300 \times 2 \times 0{,}344 \text{ MeV}$$

$$= 206 \text{ MeV}.$$



$$E_\pi = \sqrt{(cp)^2 + (m_\pi c^2)^2} = 249 \text{ MeV}$$

$$m_{K^\circ} = 2E_\pi = 498 \text{ MeV}$$

$\underline{8.8}$ Take $\epsilon^{0\beta\gamma\delta} \partial_\beta F_{\gamma\delta} =$

$$= \underbrace{\epsilon^{0123} \partial_1 F_{23} + \epsilon^{0132} \partial_1 F_{32}}_{\text{same}} +$$

$$+ \underbrace{\epsilon^{0213} \partial_2 F_{13} + \epsilon^{0231} \partial_2 F_{31}}_{\text{same}} +$$

$$+ \underbrace{\epsilon^{0312} \partial_3 F_{12} + \epsilon^{0321} \partial_3 F_{21}}_{\text{same}} =$$

$$= 2(\partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12}) =$$

$$= 2\underbrace{(\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12})}$$

Same as $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$

for $\alpha = 1$, $\beta = 2$, $\gamma = 3$

Similarly for all other combinations

So $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

$$\Longleftrightarrow \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

8.9 $\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) =$

$= \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = 0$ (a)

(b) set $A_\mu = \epsilon_\mu e^{ik\cdot x}$

$\partial^\mu A_\mu = i k^\mu \epsilon_\mu e^{ik\cdot x} = 0$

$\partial^2 A_\nu = (-ik^\mu)(-ik_\mu) A_\nu =$

$= -k^\mu k_\mu A_\nu = 0$

$\Rightarrow \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu = 0$

(c) $F_{\mu\nu} = i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu) e^{ik\cdot x}$

Let $\epsilon_\rho \to \epsilon_\rho + \alpha k_\rho$

$F_{\mu\nu} \to i(k_\mu(\epsilon_\nu + \alpha k_\nu) - k_\nu(\epsilon_\mu + \alpha k_\mu)) e^{ik}$

$= i(k_\mu \epsilon_\nu + \cancel{\alpha k_\mu k_\nu} - k_\mu \epsilon_\nu - \cancel{\alpha k_\nu k_\mu}) e^{ikx}$

$= F_{\mu\nu}$ ✓