Föreläsning II

Linjär ODE (andra ordningen): $y''(x) + \underbrace{a}_{a,b \text{ konstanter}}y'(x) + \underbrace{b}y(x) = g(x)$ $\quad y(x_0) = y_0 \quad y'(x_0) = y_1$

Ex. 1: $y'' - 3y' + 2y = 1 \quad y(0) = 1 \quad y'(0) = -2.$

Trick: $y'' - 3y' + 2y = \underbrace{(y' - 2y)'}_{z'} - \underbrace{(y' - 2y)}_{=:z} = 1 \rightsquigarrow z' - z = 1$

$z(0) = y'(0) - 2y(0) = \underline{\underline{-4}}.$

(Föreläsning I)

$\rightsquigarrow \quad z(x) = e^x \cdot z(0) + e^x \int_0^x e^{-t} \cdot 1\, dt = -4e^x + e^x(1 - e^{-x}) = -3e^x - 1.$

$\rightsquigarrow \quad y'(x) - 2y(x) = -3e^x - 1 \qquad y(0) = 1.$

(Föreläsning I)

$\rightsquigarrow \quad y(x) = e^{2x}\underbrace{y(0)}_{=1} + e^{2x}\underbrace{\int_0^x e^{-2t}(-3e^t - 1)\, dt}_{-3(1-e^{-x}) - \frac{1}{2}(1 - e^{-2x})}$

$= (1 - 3 - \frac{1}{2})e^{2x} + 3e^x + \frac{1}{2} = -\frac{5}{2}e^{2x} + 3e^x + \frac{1}{2}$

Obs: $y(0) = -\frac{5}{2} + 3 + \frac{1}{2} = 1$

$y'(0) = -5 + 3 = -2$

$y'(x) = -5e^{2x} + 3e^x \qquad y''(x) = -10e^{2x} + 3e^x$

$\rightsquigarrow \quad y''(x) - 3y'(x) + 2y(x) = -10e^{2x} + 3e^x - 3(-5e^{2x} + 3e^x) + 2\left(-\frac{5}{2}e^{2x} + 3e^x + \frac{1}{2}\right)$

$= 0 \cdot e^{2x} + 0 \cdot e^x + 1 = \underline{\underline{1}}$

Vad händer?

Idé: Skriv $y'' + ay' + by = \underbrace{(y' - r_1y)'}_{z'} - r_2\underbrace{(y' - r_1y)}_{=:z} = g$ — för några komplexa tal r_1, r_2.

$\underbrace{\qquad}_{y'' - (r_1 + r_2)y' + r_1r_2 \cdot y} \rightsquigarrow \begin{cases} a = -(r_1 + r_2) \\ b = r_1 r_2 \end{cases}$

($\Leftrightarrow$ r_1, r_2 rötter till ekv. $r^2 + ar + b = 0$ (karaktäristiska ekv.))

Reduktion till första ordningen:

$z' - r_2 z = g \quad , \quad z(x_0) = y'(x_0) - r_1 y(x_0)$

$= y_1 - r_1 y_0 =: z_0$

$\rightsquigarrow \quad z(x) = e^{r_2(x - x_0)} z_0 + e^{r_2(x - x_0)} \int_{x_0}^x e^{-r_2(t - x_0)} g(t)\, dt$

$= e^{r_2 x} \cdot e^{-r x_0} z_0 + \int_{x_0}^x e^{r_2(x - t)} g(t)\, dt =: h(x)$

- Vi kommer senare att göra reda för vad $x \longmapsto e^{rx}$ betyder då r är ett komplext tal.

$\rightsquigarrow \quad y'(x) - r_1 y(x) = h(x) \quad , \quad y(x_0) = y_0$

$\rightsquigarrow \quad y(x) = e^{r_1(x - x_0)} y_0 + e^{r_1(x - x_0)} \int_{x_0}^x e^{-r_1(t - x_0)} h(t)\, dt$

$= e^{r_1 x}(e^{-r_1 x_0} y_0) + \int_{x_0}^x e^{r_1(x - t)} h(t)\, dt$

Ex 2: $y'' + y = x$, $y(0) = 0$, $y'(0) = 1$

[Kar. ekv.] $r^2 + 1 = 0 \rightsquigarrow r_1 = i,\ r_2 = -i.$ (imaginära rötter !)

$$\rightsquigarrow \quad y'' + y = \underbrace{(y' - iy)'}_{z'} + i\,\underbrace{(y' - iy)}_{=z} = x \quad , \quad z(0) = y'(0) - iy(0) = 1.$$

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$$\rightsquigarrow \quad z(x) = e^{-ix} z(0) + e^{-ix}\int_0^x e^{it}\, t\, dt = 1 - ix$$

$$\rightsquigarrow \quad y'(x) - iy(x) = 1 - ix \ , \ y(0) = 0 \quad \rightsquigarrow \quad y(x) = e^{ix}\int_0^x e^{-it}(1 - it)\, dt = \ldots = x$$

Obs: $y(0) = 0$, $y'(0) = 1$

$y'' + y = x$

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Ex 3: $y'' - 2y' + y = 1$, $y(0) = 1$, $y'(0) = 1$

[Kar. ekv.] $\underbrace{r^2 - 2r + 1}_{=(r-1)^2} = 0 \rightsquigarrow r_1 = r_2 = 1$ (dubbelrot!)

$$\rightsquigarrow \quad y'' - 2y' + y = \underbrace{(y' - y)'}_{=z'} - \underbrace{(y' - y)}_{=z} = 1 \quad , \quad z(0) = y'(0) - y(0) = 0$$

$$\rightsquigarrow \quad z(x) = e^x \int_0^x e^{-t}\cdot 1\, dt = e^x(1 - e^{-x}) = e^x - 1.$$

$$\rightsquigarrow \quad y'(x) - y(x) = e^x - 1 \ , \ y(0) = 1 \quad \rightsquigarrow \quad y(x) = e^x + e^x \int_0^x \underbrace{e^{-t}(e^t - 1)}_{=1 - e^{-t}}\, dt$$

$$= e^x + e^x(x - 1 + e^{-x})$$

$$= xe^x + 1$$

Obs: $y(0) = 1$ $y'(0) = 1$

$y'(x) = (1 + x)e^x$, $y''(x) = e^x + (1 + x)e^x = (2 + x)e^x$

$y'' - 2y' + y = (\cancel{2} + \cancel{x})e^x - 2(\cancel{1} + \cancel{x})e^x + \cancel{xe^x} + 1 = 1.$

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Linjär ODE (högre ordning)

Ex. 4: $y''' - 4y'' + 5y' - 2y = 0$ $y(0) = y'(0) = 0$, $y''(0) = 1$

[Kar. ekv.] $r^3 - 4r^2 + 5r - 2 = (r - 2)(r^2 - 2r + 1)$ $[r_1 = 2,\ r_2 = r_3 = 1]$

$$\rightsquigarrow \quad y''' - 4y'' + 5y' - 2y = \underbrace{(y'' - 2y' + y)'}_{=z'} - 2\underbrace{(y'' - 2y' + y)}_{=:z} = 0 \quad , \quad z(0) = y''(0) - 2y'(0) + y(0) = 1$$

$$\rightsquigarrow \quad \underbrace{y'' - 2y' + y}_{\underbrace{(y' - y)'}_{=w'} - \underbrace{(y' - y)}_{=:w}} = e^{2x} \quad \rightsquigarrow \quad w(x) = e^x \int_0^x e^{-t} e^{2t}\, dt = e^x(e^x - 1) = e^{2x} - e^x$$

$w(0) = 0$

$$\rightsquigarrow \quad y' - y = e^{2x} - e^x \ , \ y(0) = 0 \quad \rightsquigarrow \quad y(x) = e^x \int_0^x e^{-t}(e^{2t} - e^t)\, dt$$

$$= e^x(e^x - 1 - x) = e^{2x} - (1 + x)e^x$$