Storgruppsövning I

Betrakta $y'(x) = f\left(\frac{y(x)}{x}\right)$, $y(x_0) = y_0$ $\quad (x_0 \neq 0,\ f(u) - u \neq 0,\ u \approx y_0/x_0)$

Idé: Låt $u = y/x \rightsquigarrow u' = \frac{xy' - y}{x^2} = \frac{x f(y/x) - y}{x^2} = \frac{1}{x}(f(u) - u)$

$$\rightsquigarrow \left(\frac{1}{f(u)-u}\right)u' = \frac{1}{x}, \quad u(x_0) = \underbrace{y_0/x_0}_{=:u_0} \qquad \text{[separabel ODE!]}$$

Sätt $Q(u) = \int_{y_0/x_0}^{u} \frac{1}{f(s)-s}\,ds$, och lös $\boxed{Q(u(x)) = \ln x - \ln x_0}$ $\quad \int_{x_0}^{x} \frac{dt}{t}$

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• Tenta 18 jan-19: 2.) Lös $\boxed{x \cdot y'(x) = y(x) + \sqrt{y(x)^2 - x^2},\ x > 0;\ y(1) = 2.}$ $(*)$

$$(*) \quad xy' = y + x\sqrt{(y/x)^2 - 1}$$

$$\iff y' = \frac{y}{x} + \sqrt{(y/x)^2 - 1} = f(y/x) \qquad f(u) = u + \sqrt{u^2 - 1}, \quad u_0 = 2$$

$$\rightsquigarrow Q(u) = \int_2^u \frac{1}{\sqrt{s^2-1}}\,ds$$

TMA972: Variabelbyte $t = s + \sqrt{s^2 - 1}$ [PB1; 5.3] $\quad s: 2+\sqrt{3} \to u + \sqrt{u^2-1}$ $\rightsquigarrow \begin{cases} \frac{1}{\sqrt{s^2-1}} = \frac{2t}{t^2-1} \\ ds = \frac{t^2-1}{2t^2} \end{cases}$

$$\rightsquigarrow Q(u) = \int_{2+\sqrt{3}}^{u+\sqrt{u^2-1}} \frac{dt}{t} = \ln\left(u + \sqrt{u^2-1}\right) - \ln(2+\sqrt{3}) \overset{?}{=} \ln x - \underbrace{\ln 1}_{=0}$$

$$\rightsquigarrow u + \sqrt{u^2 - 1} = \underbrace{(2+\sqrt{3})}_{=:c}x \qquad (\text{lös ut } u = u(x))$$

$$\Big(\cancel{u^2} - 1 = (cx - u)^2 = c^2x^2 - 2cx \cdot u + \cancel{u^2} \rightsquigarrow u(x) = \frac{c^2x^2 + 1}{2cx} = \frac{y(x)}{x}$$

$$\rightsquigarrow \boxed{y(x) = \frac{(2+\sqrt{3})^2x^2 + 1}{2(2+\sqrt{3})}}$$

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• Tenta 18 jan-19: 1) Lös $y''' - 3y'' + 4y = e^{2x} + e^{-x}$, $y(0) = y'(0) = y''(0) = 0$

Kom ihåg: $r_1r_2r_3 = -4$

[Kar. ekv.] $r^3 - 3r^2 + 4 = 0$ $\quad$ (Inspektion: $r_1 = -1$, $r_2 = 2$, $r_3 = \frac{-4}{-2} = 2$)

$$= (r+1)\underbrace{(r-2)^2}_{= r^2 - 4r + 4}$$

$$\rightsquigarrow y''' - 3y'' + 4y = \underbrace{(y'' - 4y' + 4y)'}_{=:z'} + \underbrace{(y'' - 4y' + 4y)}_{z} = e^{2x} + e^{-x}, \quad z(0) = 0$$

$$\rightsquigarrow z(x) = e^{-x}\int_0^x e^t\left(e^{2t} + e^{-t}\right)dt = e^{-x}\left(\frac{1}{3}(e^{3x} - 1) + x\right)$$

$$= \frac{1}{3}e^{2x} - \frac{1}{3}e^{-x} + xe^{-x}$$

$$\rightsquigarrow \quad \underbrace{y''-4y'+4y}_{} = \frac{1}{3}e^{2x} - \frac{1}{3}e^{-x} + x e^{-x} \quad , \quad y(0) = y'(0) = 0$$

$$(\underbrace{y'-2y}_{=w'})' - 2(\underbrace{y'-2y}_{=w}) \quad , \quad w(0) = 0$$

$$\rightsquigarrow \quad w(x) = e^{2x}\int_0^x \underbrace{e^{-2t}\left(\frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} + t e^{-t}\right)}_{= \frac{1}{3} - \frac{1}{3}e^{-3t} + t e^{-3t}} dt$$

$$= e^{2x}\left(\frac{1}{3}x - \frac{1}{9}(\cancel{1} - \cancel{e^{-3x}}) + \frac{e^{-3x}}{9}(-3x - \cancel{1}) + \cancel{\frac{1}{9}}\right)$$

$$= \frac{1}{3}\left(x e^{2x} - x e^{-x}\right)$$

$$\rightsquigarrow \quad y'(x) - 2y(x) = \frac{1}{3}x\left(e^{2x} - e^{-x}\right) \quad , \quad y(0) = 0$$

$$\rightsquigarrow \quad y(x) = e^{2x}\underbrace{\int_0^x \underbrace{e^{-2t}\,\frac{1}{3}t\left(e^{2t} - e^{-t}\right)}_{= \frac{1}{3}t - \frac{1}{3}t e^{-3t}} dt}_{}$$

$$= \frac{x^2}{6} - \frac{1}{3}\left(\frac{1}{9}e^{-3x}(-3x-1) + \frac{1}{9}\right)$$

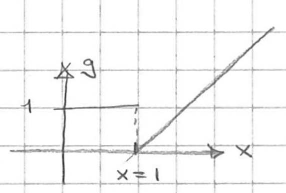
$$= \frac{x^2}{6} + \frac{x}{9}e^{-3x} + \frac{1}{27}e^{-3x} - \frac{1}{27}$$

$$\rightsquigarrow \quad \boxed{y(x) = \left(\frac{x^2}{6} - \frac{1}{27}\right)e^{2x} + \frac{1}{9}\left(x + \frac{1}{3}\right)e^{-x}}$$

——— x ———

Ex. Löse $y'(x) - y(x) = \begin{cases} 1 & x \leq 1 \\ x-1 & x > 1 \end{cases} =: g(x)$

$y(0) = 2$



$$\rightsquigarrow \quad y(x) = e^x \underbrace{y_0}_{=2} + e^x \int_0^x e^{-t} g(t)\, dt$$

$$= 2e^x + e^x \cdot \begin{cases} \int_0^x e^{-t}\, dt & x \leq 1 \quad (=: A) \\ \int_0^1 e^{-t}\, dt + \int_1^x e^{-t}(t-1)\, dt & x > 1 \quad (=: B) \end{cases}$$

$$A = 1 - e^{-x}$$

$$B = (1 - \cancel{e^{-1}}) + \underbrace{\left[e^{-t}(-t-1)\right]_1^x}_{= -e^{-x}(1+x) + \cancel{2e^{-1}}} - (\cancel{e^{-1}} - e^{-x}) = 1 - x e^{-x}$$

$$\rightsquigarrow \quad \boxed{y(x) = \begin{cases} 3e^x - 1 & 0 \leq x \leq 1 \\ 3e^x - x & x > 1 \end{cases}}$$