Föreläsning III

Sats: Antag att $f:(a,b)\to\mathbb{R}$ är $n+1$ ggr kontinuerligt deriverbar.

Fixera $x_0\in(a,b)$. Då gäller det att $\forall y\in\mathbb{R}$ så att $x_0+y\in(a,b)$

att det existerar $\theta=\theta(x_0,y)\in(0,1)$ så att

$$f(x_0+y)=\underbrace{\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}y^k}_{\text{• Taylorpolynom av grad } n \text{ tillhörande } f \text{ och } x_0.}+\underbrace{\frac{f^{(n+1)}(x_0+\theta y)}{(n+1)!}y^{n+1}}_{\text{• Lagranges restterm.}} \qquad (*)$$

(oftast mycket liten om $y\approx 0$).

• <u>Viktigt specialfall</u>: $c<0<d$, $g:(c,d)\to\mathbb{R}$ $(n+1)$ ggr kontinuerligt deriverbar: $\forall y\in(c,d)$ $\exists\,\theta=\theta(y)\in(0,1)$ s.a.

$$g(y)=\underbrace{\sum_{k=0}^{n}\frac{g^{(k)}(0)}{k!}y^k}_{\text{• Maclaurinpolynom av grad } n \text{ tillhörande } g.}+\frac{g^{(n+1)}(\theta y)}{(n+1)!}y^{n+1} \qquad (**)$$

<u>Obs</u>: $(**)\Rightarrow(*)$: $g(y)=f(x_0+y)$, $c=a-x_0$, $d=b-x_0$

$\left(\rightsquigarrow g^{(k)}(0)=f^{(k)}(x_0)\ \forall k.\right)$

<u>Tillämpningar</u>:

<u>Ex 1</u>: $g(y)=e^y$, $g^{(k)}(0)=e^y\big|_{y=0}=1\quad\forall k\geq 0$.

$\rightsquigarrow\ \forall n\geq 0\ \exists\,\theta_n(y)\in(0,1)$ så att

$$e^y=\sum_{k=0}^{n}\frac{y^k}{k!}+\frac{e^{\theta_n(y)\cdot y}}{(n+1)!}$$

I synnerhet: $[y=1]$ $\quad e=\sum_{k=0}^{n}\frac{1}{k!}+\frac{e^{\theta_n}}{(n+1)!}$, $\theta_n=\theta_n(1)\in(0,1)$

$$\rightsquigarrow\quad 0<e-\sum_{k=0}^{n}\frac{1}{k!}<\frac{e}{(n+1)!}\qquad\forall n\geq 0 \qquad (***)$$

Vill att detta leder till en motsägelse.

<u>Sats</u> [Euler 1737] e är irrationellt.

<u>Bevis</u> [Fourier] Antag att $e=\frac{p}{q}$, $p,q\geq 1$ heltal (d.v.s. e är rationell.)

$$(***)\rightsquigarrow\quad 0<\frac{p}{q}-\sum_{k=0}^{n}\frac{1}{k!}<\frac{e}{(n+1)!}\quad\forall n\geq 0 \overset{\text{mult. med } n!}{\rightsquigarrow} 0<\underbrace{\frac{pn!}{q}}_{\text{• heltal om } n\geq q.}-\underbrace{\sum_{k=0}^{n}\frac{n!}{k!}}_{\text{• heltal } \forall n}<\underbrace{\frac{e}{n+1}}_{<1\ \forall n\geq 2}$$

$\rightsquigarrow\quad 0<\text{heltal}<1$.

$\forall n\geq\max(2,q)$ $\quad\hookrightarrow$ Omöjligt! $\quad\lightning$

Ex 2: $g(y) = \ln(1+y) \rightsquigarrow g^{(1)}(y) = \frac{1}{1+y}$, $g^{(2)}(y) = \frac{-1}{(1+y)^2}$, ..., $\underbrace{g^{(n)}(y) = \frac{(-1)^{n-1}(n-1)!}{(1+y)^n}}_{\forall n \geq 1}$

$\rightsquigarrow g^{(n)}(0) = (-1)^{n-1}(n-1)! \quad \forall n \geq 1 \quad (g^{(0)}(0) = 0)$

$$\rightsquigarrow \ln(1+y) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} y^k + \frac{g^{(n+1)}(\theta_n \cdot y)}{(n+1)!} y^{n+1}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} y^k + \underbrace{\frac{(-1)^n}{(n+1)\cdot(1+\theta_n y)^{n+1}} \cdot y^{n+1}}_{|\cdot| \leq \frac{y^{n+1}}{n+1}}.$$

• Beräkna $\ln(1.1)$ med 4 decimalers noggrannhet.

$$\ln(1.1) = \ln\left(1 + \frac{1}{10}\right) = g\left(\frac{1}{10}\right)$$

4 decimalers noggrannhet

$$\left| g\left(\frac{1}{10}\right) - \sum_{k=1}^{n} \frac{(-1)^{k-1} \cdot \left(\frac{1}{10}\right)^k}{k} \right| \leq \frac{10^{-(n+1)}}{n+1} < 10^{-5} \quad \text{om } n \geq 4$$

$$\rightsquigarrow \left| \ln(1.1) - \underbrace{\left(\frac{1}{10} - \frac{1}{2} \cdot \frac{1}{10^2} + \frac{1}{3} \cdot \frac{1}{10^3} - \frac{1}{4} \cdot \frac{1}{10^4} \right)}_{= 0{,}09530833\ldots} \right| < 10^{-5}$$

$\rightsquigarrow \ln(1.1) = 0{,}0953\ldots$ skiljer sig åt i femte decimalen.

$(0{,}09531 01798\ldots)$ 10 decimalers noggrannhet.

Ex 3: Beräkna Fresnel-integralen $\int_0^1 \sin x^2 \, dx$ med 5 decimalers noggrannhet.

Idé: $g(y) = \sin y \rightsquigarrow g^{(1)}(y) = \cos y$, $g^{(2)}(y) = -\sin y$, $g^{(3)}(y) = -\cos y$, ...

$$\rightsquigarrow g^{(k)}(0) = \begin{cases} 0 & k \text{ jämn } (=2m) \\ (-1)^m & k = 2m+1 \end{cases}, \quad m \geq 0, \quad |g^{k}(t)| \leq 1 \quad \forall t \in \mathbb{R} \quad \forall k$$

$$\overset{n=2r+1}{\rightsquigarrow} \left| \sin y - \sum_{m=0}^{r} \frac{(-1)^m}{(2m+1)!} y^{2m+1} \right| \leq \frac{1}{(2m+2)!} \underbrace{|y|^{2m+2}}_{\leq 1} \quad \forall\, 0 \leq y \leq 1.$$

$$\overset{y = x^2 \in [0,1]}{\rightsquigarrow} \left| \sin x^2 - \underbrace{\sum_{m=0}^{r} \frac{(-1)^m}{(2m+1)!} x^{2(2m+1)}}_{p_r(x)} \right| \leq \frac{1}{\underbrace{(2r+2)!}_{2(r+1)}} < 10^{-6} \quad \text{om } r \geq 4$$

$(10! > 10^7)$

$$\rightsquigarrow \left| \int_0^1 \sin x^2 \, dx - \int_0^1 p_4(x) \, dx \right| < 10^{-6}$$

till 5 decimalen

$$\rightsquigarrow \int_0^1 \sin x^2 \, dx \overset{\downarrow}{=} \int_0^1 p_4(x) \, dx = \int_0^1 \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} \right) dx = 0.31026\ldots$$

Bevis av (**): Alt: $\xi = \theta(y)y$, $0 < \theta(y) < 1$.

Vill visa: $\exists\, 0 < \xi < y$ s.a. $g(y) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} y^k + \frac{g^{(n+1)}(\xi)}{(n+1)!} y^{n+1}$ (■)

fix!

- Medelvärdessatsen: Om $\varphi : [0,y] \to \mathbb{R}$, $\varphi(0) = \varphi(y)$, φ kontinuerligt derivenbar på $(0,y)$ så existerar $\xi \in (0,y)$ s.a. $\varphi'(\xi) = 0$.

Alt version: Om $h, k : [0,y] \to \mathbb{R}$ kontinuerligt derivenbara på $(0,y)$ så existerar $\xi \in (0,y)$ s.a. $(h(y) - h(0))k'(\xi) = (k(y) - k(0))h'(\xi)$.

(Bevis av alt. version: Låt $\varphi(x) = (h(y) - h(0))k(x) - (k(y) - k(0))h(x)$.

Då gäller $\varphi(0) = (h(y) - h(0))k(0) - (k(y) - k(0))h(0) = h(y)k(0) - k(y)h(0)$

$\varphi(y) = (h(y) - h(0))k(y) - (k(y) - k(0))h(y) = -k(y)h(0) + k(0)h(y)$

$\rightsquigarrow \exists\, \xi \in (0,y)$ s.a. $\varphi'(\xi) = (h(y) - h(0))k'(\xi) - (k(y) - k(0))h'(\xi)$ ⚡)

Bevis av (■):

Sätt $h_y(x) = \sum_{k=0}^{n} \frac{g^{(k)}(x)}{k!}(y-x)^k \rightsquigarrow h_y(0) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} y^k$, $h_y(y) = g(y)$

$k_y(x) = (y-x)^{n+1} \rightsquigarrow k_y(0) = y^{n+1}$, $k_y(y) = 0$

Vi har även: $h_y'(x) = \sum_{k=0}^{n} \frac{1}{k!}\left(g^{(k+1)}(x)(y-x)^k - g^{(k)}(x) \cdot k\,(y-x)^{k-1}\right)$

$$= \sum_{k=0}^{n} \frac{g^{(k+1)}(x)(y-x)^k}{k!} - \underbrace{\sum_{k=1}^{n} \frac{g^{(k)}(x)}{(k-1)!}(y-x)^{k-1}}_{= \sum_{k=0}^{n-1} \frac{g^{(k+1)}(x)}{k!}(y-x)^k} = \frac{g^{(n+1)}(x)(y-x)^n}{n!}$$

$\rightsquigarrow k_y'(x) = -(n+1)(y-x)^n$

Alt. version av Medelvärdessatsen: $\exists\, 0 < \xi < y$ s.a.

$$\underbrace{(h_y(y) - h_y(0))k_y'(\xi)}_{\cancel{-}\left(g(y) - \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} y^k\right)(n+1)\underbrace{\cancel{(y-\xi)^n}}_{\neq 0}} = \underbrace{(k_y(y) - k_y(0))h_y'(\xi)}_{\cancel{-}y^{n+1} \cdot \frac{g^{(n+1)}(\xi)}{n!}\underbrace{\cancel{(y-\xi)^n}}_{\neq 0}}$$

$$\rightsquigarrow g(y) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} y^k + \frac{g^{(n+1)}(\xi)}{(n+1)!} y^{n+1}$$ ⚡