Föreläsning VI

Linjär DE (första ordningen)

- $y_{n+1} = p_n y_n + q_n, \quad \forall n \geq 0.$ (y_0 antas vara känd).

<u>Obs</u>: $y_1 = p_0 y_0 + q_0, \quad y_2 = p_1(p_0 y_0 + q_0) + q_1 = p_1 p_0 \cdot y_0 + p_1 q_0 + q_1.$

$$y_3 = p_2(p_1 p_0 \cdot y_0 + p_1 q_0 + q_1) + q_2 = p_2 p_1 p_0 \cdot y_0 + p_2 p_1 q_0 + p_2 q_1 + q_2.$$

$$y_4 = p_3 y_3 + q_3 = p_3 p_2 p_1 p_0 \cdot y_0 + p_3 p_2 p_1 q_0 + p_3 p_2 q_1 + p_3 q_2 + q_3.$$

$$\rightsquigarrow \quad y_{n+1} = p_n \cdots p_0 \cdot y_0 + \Big(\underbrace{p_n \cdots p_1}_{=:P_{n,0}} q_0 + \underbrace{p_n \cdots p_2}_{=:P_{n,1}} q_1 + \dots + \underbrace{p_n}_{=:P_{n,n-1}} q_{n-1} + q_n\Big) \qquad (P_{n,n} := 1)$$

$$= p_n \cdots p_0 \cdot y_0 + \sum_{k=0}^{n} P_{n,k}\, q_k \quad \forall n \geq 0.$$

<u>Viktigt specialfall</u>: $p_n = r \quad \forall n \geq 0 \qquad y_{n+1} - r y_n = q_n$

$$\rightsquigarrow \quad \boxed{y_{n+1} = r^{n+1} y_0 + \sum_{k=0}^{n} r^{n-k} q_k} \quad \forall n \geq 0. \qquad (*)$$

<u>Ex.</u> Beräkna y_{57} om $y_{n+1} - \overset{=r}{3}\, y_n = n, \quad y_0 = -2$

$$\overset{(*)}{\rightsquigarrow} \quad y_{n+1} = 3^{n+1} \cdot (-2) + \sum_{k=0}^{n} 3^{n-k} k = -2 \cdot 3^{n+1} + 3^n \underbrace{\sum_{k=0}^{n} 3^{-k} \cdot k}_{= f(1/3)}$$

där $f(x) = \sum_{k=0}^{n} k \cdot x^k = x \cdot \frac{d}{dx}\left(\sum_{k=0}^{n} x^k\right) = x \frac{d}{dx}\left(\frac{x^{n+1}-1}{x-1}\right)$

$$= x\left(\frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2}\right)$$

$$= \frac{x}{(x-1)^2}\left(n \cdot x^{n+1} - (n+1)x^n + 1\right) \rightsquigarrow f(1/3) = \frac{3}{4}\left(n\, 3^{-(n+1)} - (n+1)3^{-n} + 1\right)$$

$$\rightsquigarrow \quad y_{n+1} = -2 \cdot 3^{n+1} + \frac{3}{4}\left(n \cdot 3^{-1} - (n+1) + 3^n\right)$$

$$= \left(-2 + \frac{1}{4}\right)3^{n+1} + \frac{1}{4}\left(n - 3(n+1)\right) = -\frac{7}{4} \cdot 3^{n+1} - \frac{1}{4}(2n+3)$$

$$\rightsquigarrow \quad \boxed{y_{57} = -\frac{7}{4} \cdot 3^{57} - \frac{1}{4}(2 \cdot 56 + 3)}$$

Linjär DE (andra ordningen)

- $y_{n+2} + a y_{n+1} + b y_n = q_n$ (y_0, y_1 antas vara kända)

Idé: Skriv $y_{n+2} + a y_{n+1} + b y_n = (\underbrace{y_{n+2} - r_1 y_{n+1}}_{z_{n+1}}) - r_2 (\underbrace{y_{n+1} - r_1 y_n}_{z_n}) = q_n$,

$$y_{n+2} - \underbrace{(r_1+r_2)}_{=a} y_{n+1} + \underbrace{r_1 r_2}_{=b} y_n$$

för några $r_1, r_2 \in \mathbb{C}$.

Dessa måste uppfylla: $\begin{cases} a = -(r_1+r_2) \\ b = r_1 \cdot r_2 \end{cases}$

$\rightsquigarrow$ r_1, r_2 rötter till den karaktäristiska ekv. $r^2 + ar + b = 0$.

$\rightsquigarrow$ $z_{n+1} - r_2 z_n = q_n$, $z_0 = y_1 - r_1 y_0$

$\overset{(*)}{\rightsquigarrow}_{\forall n \geq 0}$ $z_{n+1} = r_2^{n+1} z_0 + \sum_{k=0}^{n} r_2^{n-k} q_k$ $\rightsquigarrow$ $y_{n+1} - r_1 y_n = z_n$

$$\overset{(*)}{\rightsquigarrow} \quad \boxed{y_{n+1} = r_1^{n+1} y_0 + \sum_{l=0}^{n} r_1^{n-l} z_l} \qquad (**)$$

Ex. Lös $y_{n+2} - 3y_{n+1} + 2y_n = \frac{(-1)^n}{n+1}$, $y_0 = y_1 = 0$.

[Kar. ekv] $r^2 - 3r + 2 = 0 \rightsquigarrow r_1 = 1,\ r_2 = 2$.

Skriv: $y_{n+2} - 3y_{n+1} + 2y_n = (\underbrace{y_{n+2} - y_{n+1}}_{= z_{n+1}}) - 2(\underbrace{y_{n+1} - y_n}_{= z_n}) = \frac{(-1)^n}{n+1}$, $z_0 = 0$

$$\overset{(*)}{\rightsquigarrow} z_{n+1} = \sum_{k=0}^{n} 2^{n-k} \frac{(-1)^k}{k+1} = 2^n \sum_{k=0}^{n} \frac{(-2)^k}{k+1}$$

$$\rightsquigarrow z_n = 2^{n-1} \sum_{k=0}^{n-1} \frac{(-2)^k}{k+1} \overset{(*)}{\rightsquigarrow} y_{n+1} = \sum_{l=0}^{n} z_l = \boxed{\sum_{l=0}^{n} 2^{l-1} \sum_{k=0}^{l-1} \frac{(-2)^k}{k+1}}$$

$(z_0 = 0)$

Ex. Beräkna y_{17} om $y_{n+2} - 2y_{n+1} + y_n = 1$, $y_0 = 1$, $y_1 = -1$

[Kar. ekv] $r^2 - 2r + 1 = 0 \rightsquigarrow r_1 = r_2 = 1$.

Skriv: $y_{n+2} - 2y_{n+1} + y_n = (\underbrace{y_{n+2} - y_{n+1}}_{= z_{n+1}}) - (\underbrace{y_{n+1} - y_n}_{= z_n}) = 1$, $z_0 = y_1 - y_0 = \underline{\underline{-2}}$

$$\overset{(*)}{\rightsquigarrow} z_{n+1} = -2 + \sum_{k=0}^{n} 1 = -2 + n + 1 = n - 1 \rightsquigarrow z_n = n - 2$$

$$\rightsquigarrow y_{n+1} - y_n = n - 2 \overset{(*)}{\rightsquigarrow} y_{n+1} = 1 + \sum_{l=0}^{n} (l - 2) = 1 + \frac{n(n+1)}{2} - 2(n+1)$$

$$\rightsquigarrow y_n = 1 + \frac{n(n-1)}{2} - 2n = \boxed{\frac{1}{2}(n^2 - 5n + 2)}$$

Kontroll: $y_0 = 1$, $y_1 = -1$

$y_{n+2} - 2y_{n+1} + y_n$

$= \frac{1}{2}\left(\left((n+2)^2 - 5(n+2) + 2\right) - 2\left((n+1)^2 - 5(n+1) + 2\right) + n^2 - 5n + 2\right) = 1$

Viktigt specialfall: $g_n = 0 \ \forall n \geq 0$ och $r_1 \neq r_2$ viktigt!

$\overset{(**)}{\rightsquigarrow}$ $y_n = A \cdot r_1^n + B r_2^n$ för några konstanter A, B.
(som kan bestämmas från y_0 o y_1

övning!

Ex. Beräkna $D_n = \det \begin{pmatrix} 2\cos\alpha & 1 & 0 & \cdots & \\ 1 & 2\cos\alpha & 1 & 0 & \cdots \\ & 1 & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \end{pmatrix}$ för $n \geq 1$.

Ko-faktoruppdelning: $D_{n+2} = 2\cos\alpha \cdot D_{n+1} - 1 \cdot D_n \quad \forall n \geq 1$. (•)

Obs: $D_1 = 2\cos\alpha \qquad D_2 = 4\cos^2\alpha - 1$ — sätt $D_0 = 0$

$\rightsquigarrow \quad D_2 = 2\cos\alpha \cdot D_1 - D_0$; så (•) även uppfyllt då $n = 0$.

[Kar. ekv.] $r^2 - 2\cos\alpha \cdot r + 1 = 0$ (Antag $-\pi/2 \leq \alpha \leq \pi/2$ så att $\sin\alpha \geq 0$)

$\rightsquigarrow \quad r_1 = e^{i\alpha} \qquad r_2 = e^{-i\alpha}$

$\rightsquigarrow \quad D_n = A e^{in\alpha} + B e^{-in\alpha} \quad \forall n \geq 0$

$D_0 = A + B = 0 \rightsquigarrow B = -A$.

$D_1 = A(e^{i\alpha} - e^{-i\alpha}) = 2iA\sin\alpha = 2\cos\alpha \rightsquigarrow A = \frac{1}{i}\cot\alpha$

$\rightsquigarrow \quad D_n = \frac{1}{i}\cot\alpha \cdot (e^{i\alpha n} - e^{-i\alpha n}) = \boxed{2\cot\alpha \cdot \sin n\alpha} \quad \forall n \geq 0$.

Kontroll: $D_0 = 0 \qquad D_1 = 2\cot\alpha \cdot \sin\alpha = 2\cos\alpha$

$$D_{n+2} - 2\cos\alpha \cdot D_{n+1} + D_n = 2\cot\alpha\,\underbrace{\Big(\underbrace{\sin(n+2)\alpha}_{\substack{\sin n\alpha \cdot \cos 2\alpha \\ +\cos n\alpha \cdot \sin 2\alpha}} - 2\cos\alpha \cdot \underbrace{\sin(n+1)\alpha}_{\substack{\sin n\alpha \cdot \cos\alpha \\ +\cos n\alpha \cdot \sin\alpha}} + \sin n\alpha\Big)}$$

$= (\cos 2\alpha - 2\cos\alpha + 1)\sin n\alpha$
$+ (\sin 2\alpha - 2\cos\alpha\sin\alpha)\cos n\alpha = 0$

$\forall n \geq 0$.