Storgruppsövning III

Variation av integralkriteriet:

- $p_1, p_2, \ldots$ positiv talföljd, s.a. $t_n = p_1 + \ldots + p_n \to \infty$, då $n \to \infty$.
- $f : [p_1, \infty) \to [0, \infty)$ monotont avtagande, $\int_{a_1}^{\infty} f(x)\,dx < +\infty$.

Då gäller det att $\sum_{k=1}^{\infty} p_k f(t_k)$ konvergent. $(*)$

Bevis: $p_k = t_k - t_{k-1}$, $\forall k \geq 1$ (sätt $t_0 = 0$)

$$\leadsto \quad s_n = \sum_{k=1}^{n} p_k f(t_k) = p_1 f(t_1) + \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \underbrace{f(t_k)}_{\leq f(x)} dx$$

$$\leq p_1 f(p_1) + \underbrace{\sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} f(x)\,dx}$$

$$\int_{t_1}^{t_n} f(x)\,dx \leq \int_{p_1}^{\infty} f(x)\,dx \quad \forall n$$

$$\leq p_1 f(p_1) + \int_{p_1}^{\infty} f(x)\,dx < +\infty \quad \leadsto \quad (s_n) \text{ begränsad}$$

Då $p_k \geq 0$ $\forall k$

$\Longrightarrow$ $s = \lim_n s_n$ existerar.

Tenta 17 Aug - 2017 6) Antag att $t_n := p_1 + \ldots + p_n > 1$ $\forall n \geq 1$, och att (t_n) är strängt växande.

Visa att $\sum_{n=1}^{\infty} \dfrac{p_n}{t_n (\ln t_n)^2}$ konvergerar.

Obs. $p_1 > 1$, $p_n = \underbrace{t_n - t_{n-1}}_{\geq 0}$ $\forall n \geq 2$

Sätt $f(x) = \dfrac{1}{x (\ln x)^2}$, $x \geq p_1$; monotont avt.

$$\int_{p_1}^{\infty} \frac{dx}{x (\ln x)^2} = \left\{ \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array} \right\} = \int_{\ln p_1}^{\infty} \frac{du}{u^2} = \frac{1}{\ln p_1} < +\infty$$

$(*) \leadsto \sum_{n=1}^{\infty} \dfrac{p_n}{t_n (\ln t_n)^2}$ konvergerar.

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Tenta 11 april 2017 4) Beräkna $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1+\frac{k}{n^2}} - 1 \right)$

Taylorutveckla: $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$.

$\rightsquigarrow \sqrt{1+\frac{k}{n^2}} - 1 = \frac{1}{2}\cdot\frac{k}{n^2} + \underbrace{O\left(\frac{k^2}{n^4}\right)}_{= O\left(\frac{1}{n^2}\right)} \quad \forall k \leq n.$

$\rightsquigarrow \sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}} - 1\right) = \underbrace{\frac{1}{2n^2} \underbrace{\sum_{k=1}^{n} k}_{= \frac{1}{2}n(n+1)}}_{} + \underbrace{n \cdot O\left(\frac{1}{n^2}\right)}_{= O\left(\frac{1}{n}\right) \to 0,\ n\to\infty}$

$\frac{1}{4} \cdot \frac{n(n+1)}{n^2} \longrightarrow \frac{1}{4}$, då $n \to \infty$.

$\rightsquigarrow \lim_{n\to\infty} \sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}} - 1\right) = \frac{1}{4}$.

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Tenta 23 Aug 2018 4) Givet $a, b, p \in \mathbb{R}$, betrakta

$$y_{n+1} = p y_{n-1} + (1-p) y_n, \quad y_1 = a, \ y_2 = b$$

för $n \geq 1$.

För vilka a, b, p existerar gränsvärdet $\lim_{n\to\infty} y_n$?

[Kar. ekv.] $t^2 - (1-p)t - p = 0$

$t = \frac{1}{2}\left((1-p) \pm \sqrt{\underbrace{(1-p)^2 + 4p}_{= 1+2p+p^2 = (1+p)^2}}\right) = \frac{1}{2}\left((1-p) \pm |1+p|\right)$

$1-p+(1+p)$

Tre fall:

1) $p > -1$: $\overbrace{t_1 = 1 \quad t_2 = -p}^{\text{olika rötter!}} \rightsquigarrow y_n = A + B(-p)^n$

Om $|p|<1 \rightsquigarrow \lim_n y_n = A$ (för alla $a, b \in \mathbb{R}$)

- $p \geq 1 \rightsquigarrow \lim_n y_n = A$ (om $B = 0$) — annars existerar ej gränsv.

($y_1 = A - pB = a$, $y_2 = A + p^2B = b \rightsquigarrow B = \frac{b-a}{p(1+p)} = 0 \Leftrightarrow \underline{b = a}$

2) $p = -1$: $t_1 = t_2 = 1 \rightsquigarrow y_n = A + Bn$ ($\lim_{n\to\infty} y_n$ exist $\Leftrightarrow B = 0$)

$y_1 = A + B = a$, $y_2 = A + 2B = b \rightsquigarrow B = b - a = 0 \Leftrightarrow \underline{b = a}$

3) $p < -1$: $t_1 = -p$, $t_2 = 1 \rightsquigarrow y_n = A + B(-p)^n$

$\rightsquigarrow \lim_n y_n$ existerar $\Leftrightarrow B = 0 \Leftrightarrow b = a$.

Tenta 12 januari 2018 : 4)

Beräkna $\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p$ (för $p>0$)

Obs [Jmf. integralkriteriet] sätt $f(x) = x^p$, $x \geq 0$, $p > 0$

$\rightsquigarrow$ $\forall k \geq 1$ gäller $k^p \leq \int_k^{k+1} f(x)\,dx \leq (k+1)^p$

$$\rightsquigarrow \underbrace{\sum_{k=1}^{n} k^p}_{=: s_n} \leq \underbrace{\sum_{k=1}^{n} \int_k^{k+1} f(x)\,dx}_{= \int_1^{n+1} x^p\,dx = \frac{1}{p+1}\left((n+1)^p - 1\right)} \leq \sum_{k=1}^{n} (k+1)^p = \underbrace{\sum_{k=1}^{n+1} k^p - 1}_{s_{n+1} - 1}$$

$$\rightsquigarrow s_n \leq \frac{1}{p+1}\left((n+1)^p - 1\right) \leq s_{n+1} - 1 \quad \forall n \geq 1$$

$s_n \geq 1 + \frac{1}{p+1}(n^p - 1)$

$$\rightsquigarrow \overline{\lim_n} \frac{s_n}{n^p} \leq \frac{1}{p+1} \overline{\lim_n} \underbrace{\left(\frac{(n+1)^p - 1}{n^p}\right)}_{\to 1} \leq \frac{1}{p+1}.$$

$$\underline{\lim_n} \frac{s_n}{n^p} \geq \lim_n \left(\frac{1}{n^p} + \frac{1}{p+1} \frac{(n^p - 1)}{n^p}\right) = \frac{1}{p+1}$$

$$\rightsquigarrow \boxed{\lim_{n\to\infty} \frac{s_n}{n} = \frac{1}{p+1}}$$