Lösningar modellexamen:

1. $-\ln\cos y = \frac{x^2}{2} + C \rightsquigarrow \cos y(x) = De^{-x^2/2}$ $\quad \left(D = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right)$

$$\rightsquigarrow \boxed{y(x) = \arccos\left(\frac{e^{-x^2/2}}{\sqrt{2}}\right), \quad x \geq 0.}$$

2. $y'' - 4y' + 4y = (\underbrace{y' - 2y}_{=z})' - 2(\underbrace{y' - 2y}_{=z}) = \sqrt{x}$ $\quad / \quad z(0) = 0$

$$\rightsquigarrow z(x) = e^{2x}\int_0^x e^{-2z}\sqrt{z}\,dz = y' - 2y, \quad y(0) = 0$$

$$\rightsquigarrow y(x) = e^{2x}\int_0^x e^{-2t}z(t)\,dt = e^{2x}\int_0^x \Big(\underbrace{\int_0^t e^{-2z}\sqrt{z}\,dz}_{=f(t)}\Big)dt$$

$$\rightsquigarrow \frac{y(x)}{xe^{2x}} = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0$$

Obs: $\lim_{t\to\infty} f(t) = \int_0^\infty e^{-2z}\sqrt{z}\,dz = \{u = 2z,\ du = 2\,dz\}$

$$= \int_0^\infty e^{-u}\sqrt{\frac{u}{2}}\cdot\frac{du}{2} = \frac{1}{2\sqrt{2}}\int_0^\infty e^{-u}\sqrt{u}\,du \overset{\text{givet!}}{=} \frac{\pi}{4\sqrt{2}}$$

$$\frac{\pi}{4\sqrt{2}} - f(t) = \int_t^\infty \underbrace{e^{-2z}\sqrt{z}}_{\leq e^{-z}\ \forall z \geq t \geq 1}\,dz \leq e^{-t} \quad \text{för } t \geq 1$$

$$\rightsquigarrow \frac{y(x)}{xe^{2x}} = \underbrace{\frac{1}{x}\int_0^x \Big(f(t) - \frac{\pi}{4\sqrt{2}}\Big)dt} + \frac{\pi}{4\sqrt{2}}$$

$$|\cdot| \leq \frac{1}{x}\underbrace{\int_0^x \Big|f(t) - \frac{\pi}{4\sqrt{2}}\Big|\,dt} \to 0, \quad \text{då } x \to \infty.$$

$$\leq \underbrace{\int_0^1}_{<+\infty} + \underbrace{\int_1^\infty e^{-t}\,dt}_{<+\infty}$$

$$\rightsquigarrow \boxed{\lim_{x\to\infty}\frac{y(x)}{xe^{2x}} = \frac{\pi}{4\sqrt{2}}}$$

3. $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + O(x^6)$, $\quad \cos 2x = 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + O(x^6)$ $\quad \left(\frac{16}{24} = \frac{4\cdot 4}{4\cdot 6} = \frac{2}{3}\right)$

$x\sin x = x^2 - \frac{x^4}{6} + O(x^6)$

$$\rightsquigarrow e^{-x^2} - \cos 2x - x\sin 3x = \cancel{1} - \cancel{x^2} + \frac{x^4}{2} - (\cancel{1} - \cancel{2x^2} + \frac{2}{3}x^4)$$

$$-(\cancel{x^2} - \frac{x^4}{6}) + O(x^6) = \Big(\underbrace{\frac{1}{2} - \frac{2}{3}}_{=-\frac{1}{6}} + \frac{1}{6}\Big)x^4 + O(x^6) = O(x^6)$$

$$\rightsquigarrow \boxed{\frac{1}{x^4}\left(e^{-x^2} - \cos 2x - x\sin 3x\right) \to 0}, \quad \text{då } x \to 0.$$

4. $a_m = 0$ om $m \neq 2^n$ för ngt $n \geq 1$

$$a_{2^n} = \frac{(-1)^n}{\ln(2+n)}, \quad n \geq 1.$$

Rotkriteriet: $$|a_m|^{1/m} = \begin{cases} 0 & m \neq 2^n \\ \underbrace{\left(\frac{1}{\ln(2+n)}\right)^{1/2^n}}_{\to 1} & m = 2^n \end{cases}$$

⇝ $\boxed{\text{Absolutkonvergent } \forall |x| < 1}$ (Divergerar $\forall |x| > 1$)

$x = \pm 1$: $F(\pm 1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(2+n)}$ $\boxed{\text{betingat konvergent (Leibniz) för } x = \pm 1}$

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5. $$y_{n+1}(x) - y_n(x) = \frac{1}{1+n^2x^2}, \quad y_0(x) = 0$$

⇝ $$y_n(x) = \sum_{k=0}^{n-1} \frac{1}{1+(kx)^2}, \quad n \geq 1, \ \forall x \in [0,1].$$

⇝ $$\frac{1}{n} y_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1+(kx)^2}$$

Obs: $x = 0$: $\frac{1}{n} y_n(0) = 1 \to 1$

$x > 0$: $\frac{1}{n} y_n(x) \leq \frac{1}{n} \underbrace{\left(\sum_{k=0}^{\infty} \frac{1}{1+(kx)^2}\right)}_{< +\infty} \to 0$

⇝ $$\lim_{n\to\infty} \underbrace{\frac{1}{n} y_n(x)}_{\text{kontinuerliga fkn}} = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$$ ~~ej~~ kontinuerlig!

$\Rightarrow$ $\boxed{\text{a) Konvergens ej likformig på } [0,1]}$

b) $$\frac{1}{n} y_n\left(\frac{1}{n}\right) = \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1+\left(\frac{k}{n}\right)^2}}_{\text{Riemannsumma!}} \to \int_0^1 \frac{dx}{1+x^2} = \left[\arctan x\right]_0^1 = \underline{\underline{\frac{\pi}{4}}}.$$

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6. $$\underbrace{f_t(x)}_{\text{avtagande}} = \frac{1}{1+\sqrt{tx}}; \quad s_n(t) := \sum_{k=1}^{n} f_t(k), \quad n \geq 1.$$

Obs: $f_t(k) \leq \int_{k-1}^{k} f_t(x)\,dx \leq f_t(k-1) \quad \forall k \geq 1$

⇝ $$s_n(t) \leq \int_0^n f_t(x)\,dx \leq \sum_{k=0}^{n-1} f_t(k) = 1 + s_n(t) - f_t(n)$$

⇝ $$0 \leq \underbrace{\int_0^n f_t(x)\,dx}_{} - s_n(t) \leq 1 - f_t(n) \leq 1 \quad \forall n \geq 1, \ \forall t > 0 \quad (*)$$

$$= \int_0^n \frac{dx}{1+\sqrt{tx}} = \frac{1}{t} \int_0^{tn} \frac{dx}{1+\sqrt{x}}$$

$$\frac{s_n(t)}{\sqrt{n}} - \frac{2}{\sqrt{t}} = \frac{1}{\sqrt{n}}\left(s_n(t) - \frac{1}{t}\int_0^{tn} \frac{dx}{1+\sqrt{x}}\right)$$

$$+ \frac{1}{\sqrt{n}\,t}\int_0^{nt}\frac{dx}{1+\sqrt{x}} - \frac{2}{\sqrt{t}} = \left\{ \frac{2}{\sqrt{t}} = \frac{1}{t\sqrt{n}} \overbrace{\int_0^{nt}\frac{du}{\sqrt{u}}}^{= 2\sqrt{nt}},\ t>0 \right\}$$

$$= \frac{1}{\sqrt{n}}\left(s_n(t) - \frac{1}{t}\int_0^{tn}\frac{dx}{1+\sqrt{x}}\right) \qquad (=A)$$

$$+ \frac{1}{t\sqrt{n}}\int_0^{tn} \underbrace{\left(\frac{1}{1+\sqrt{x}} - \frac{1}{\sqrt{x}}\right)}_{= -\frac{1}{\sqrt{x}(1+\sqrt{x})}} dx \qquad (=B)$$

$(*) \Rightarrow \quad |A| \leq 1/\sqrt{n} \to 0 \quad$ (likformigt $\forall t > 0$)

$$|B| \leq \frac{1}{t\sqrt{n}} \underbrace{\int_0^{tn} \frac{dx}{\sqrt{x}(1+\sqrt{x})}}$$

fixt $t>0$: $\underbrace{\int_0^1}_{<+\infty} + \int_1^{tn} \leq \int_1^{tn} \frac{dx}{x} = \ln(tn)$

$$\rightsquigarrow \quad |B| \leq \underbrace{\frac{1}{t\sqrt{n}}\int_0^1 + \frac{\ln tn}{t\sqrt{n}}}_{\text{likformigt om } t>1.} \to 0 \quad \rightsquigarrow \quad \boxed{\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{1+\sqrt{tk}} = \frac{2}{\sqrt{t}}}$$

$$\left(\frac{\ln tn}{t\sqrt{n}} = \underbrace{\frac{1}{\sqrt{n}} \cdot \frac{\ln t}{t}}_{\substack{\to 0 \\ \text{likf. i } t \geq 1}} + \underbrace{\left(\frac{\ln n}{\sqrt{n}}\right)\frac{1}{t}}_{\substack{\to 0 \\ (\text{likf. i } t\geq 1}} \right.$$

7. $\dfrac{\sin\overbrace{(2x+y)}^{=u}\sin\overbrace{(x+y)}^{=v}}{2x^2+y^2+3xy}$ $\qquad u = 2x+y \quad v = x+y$

$(x,y)\to(0,0) \Leftrightarrow (u,v)\to(0,0)$

$$= \frac{\sin u}{u} \cdot \frac{\sin v}{v} \cdot \underbrace{\left(\frac{u\cdot v}{2x^2+y^2+3xy}\right)} = \left(\frac{\sin u}{u}\right)\left(\frac{\sin v}{v}\right)$$

$$= \frac{(2x+y)(x+y)}{2x^2+y^2+3xy} = \frac{2x^2+y^2+3xy}{2x^2+y^2+3xy} = 1$$

$$\boxed{\lim_{(u,v)\to(0,0)} \frac{\sin u}{u}\,\frac{\sin v}{v} = 1}$$