Storgruppsövning IV

Tenta 21 Aug-14:

6) Beräkna $\lim_{n\to\infty} n\cdot\int_0^\infty \frac{\sin nx}{1+x^2}\,dx$

Antag: i) $\lim_{x\to\infty} \varphi(x) =$ ii) $\lim_{x\to\infty}\varphi'(x) = 0$ iii) $\int_0^\infty |\varphi''|\,dx < +\infty$

Mer allmänt: $\varphi : [0,\infty) \to \mathbb{R}$ 2 ggr kont. deriverbar

$$n\int_0^\infty \varphi(x)\sin nx\,dx = n\Big(\underbrace{\Big[-\varphi(x)\frac{\cos nx}{n}\Big]_0^\infty}_{=\frac{\varphi(0)}{n}} + \frac{1}{n}\int_0^\infty \varphi'(x)\cos nx\,dx\Big)$$

$$= \varphi(0) + \int_0^\infty \varphi'(x)\cos nx\,dx = \varphi(0) + \underbrace{\Big[\varphi'(x)\frac{\sin nx}{n}\Big]_0^\infty}_{=0}$$

$$-\frac{1}{n}\int_0^\infty \varphi''(x)\sin nx\,dx = \varphi(0) - \frac{1}{n}\underbrace{\int_0^\infty \varphi''(x)\sin nx\,dx}_{|\cdot| \le \int_0^\infty |\varphi''(x)|\,dx < +\infty} \to \underline{\underline{\varphi(0)}}$$

$$\rightsquigarrow \lim_{n\to\infty} n\cdot\int_0^\infty \varphi(x)\sin nx\,dx = \varphi(0)$$ (om φ uppfyller i) – iii))

I vårt fall: $\varphi(x) = \frac{1}{1+x^2}$ $\Big(\to 0$ då $x\to\infty\Big)$ $\Big(\varphi'(x) = \frac{-2x}{(1+x^2)^2} \to 0$, då $x\to\infty$

$$\varphi''(x) = -\frac{2(1+x^2)^2 + 2x\cdot 4x(1+x^2)}{(1+x^2)^4}$$

$\sim \frac{1}{x^6}$ $x \gg 1 \rightsquigarrow \int_0^\infty |\varphi''|\,dx < +\infty$

$$\rightsquigarrow \boxed{\lim_{n\to\infty} n\int_0^\infty \frac{\sin nx}{1+x^2}\,dx = 1}$$

(Bonus: Beräkna $\lim_{n\to\infty} n\int_0^\infty \underbrace{\frac{e^{-x^3\sin x}}{(2+x^4)^6}}_{\varphi(x)} \sin nx\,dx$.

$\Big[= \varphi(0) = \underline{\underline{2^{-6}}}.\Big]$

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Tenta 20 Aug-15:

4b) För vilka $x \in \mathbb{R}$ konvergerar $\sum_{n=1}^\infty \left(1+\frac{x}{n}\right)^{n^2}\cdot |x|^n$?

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Repetition: Om $a_1, a_2, \ldots$ är en talföljd, så gäller

$\lim_{n\to\infty} |a_n|^{1/n} < 1 \rightsquigarrow \sum_{n=1}^\infty a_n$ konvergerar

$\lim_{n\to\infty} |a_n|^{1/n} > 1 \rightsquigarrow \sum_{n=1}^\infty a_n$ divergerar

$\Big(\lim_{n\to\infty} |a_n|^{1/n} = 1$; ingen slutsats!$\Big)$

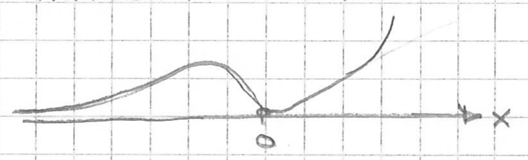
Fixera $x \in \mathbb{R}$,

$$a_n = \left(1+\frac{x}{n}\right)^{n^2} |x|^n \rightsquigarrow |a_n|^{1/n} = \underbrace{\left|1+\frac{x}{n}\right|^n}_{\to e^x} |x| \to |x|e^x$$

Om $|x|e^x < 1 \rightsquigarrow$ konvergent.

$|x|e^x > 1 \rightsquigarrow$ divergent.

Vad händer om $|x|e^x = 1$?



- $x \leq 0$: $x = -t$, $f(t) = te^{-t}$ — Maxvärde?

$$f'(t) = (1-t)e^{-t} = 0 \rightsquigarrow t = 1$$

$\max_{t \geq 0} f(t) = f(1) = e^{-1} < 1$. $\rightsquigarrow$ Ingen lösning till $|x|e^x = 1$ för $x \leq 0$

- $x \geq 0$: $x \mapsto xe^x$ växande, $\to \infty$ då $x \to \infty$

$\rightsquigarrow$ ∃unikt $x_0 \geq 0$ s.a. $x_0 e^{x_0} = 1$. $\iff$ $\log x_0 + x_0 = 0$

$$\left(1+\frac{x_0}{n}\right)^{n^2} |x_0|^n \underset{x_0 > 0}{=} e^{n^2 \ln\left(1+\frac{x_0}{n}\right) + n \log x_0} = \left\{ \ln(1+y) = y - y^2/2 + O(y^3) \right\}$$

$$= e^{n^2\left(\frac{x_0}{n} - \frac{x_0^2}{2n^2} + O\left(\frac{x_0}{n^3}\right)\right) + n\log x_0}$$

$$= e^{n\underbrace{(x_0 + \log x_0)}_{=0} - \frac{x_0^2}{2} + \underbrace{O\left(\frac{x_0}{n}\right)}_{\to 0}} \to e^{-x_0^2/2} \neq 0$$

$\rightsquigarrow \lim_n a_n \neq 0$

$\rightsquigarrow$ Serien divergerar för $x = x_0$

Tenta 22 Aug-14 :

2) Existerar gränsvärdet

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + 2y^2 + 2z^2 - 2yz}{2x^2 + 2y^2 + z^2 - 2xy} \ ?$$

Undersök längs olika linjer:

i) $y = z = 0$: $\frac{x^2}{2x^2} = \frac{1}{2} \to \frac{1}{2}$ då $x \to 0$

$\neq$ $\rightsquigarrow$ Gränsvärdet existerar ej

ii) $x = y = 0$: $\frac{2z^2}{z^2} = 2 \to 2$ då $z \to 0$

Tenta 5 April - 18 :

6. $g(x) = e^x$, $f_n(x) = \ln\left(\frac{nx}{n+1}\right)$, $x > 0$.

Avgör huruvida:

i) (f_n) konvergerar likformigt på $(0, \infty)$

ii) (g_n) konvergerar likformigt på $(0, \infty)$

• $\beta_n = \frac{n}{n+1}$, $f_n(x) = \ln \beta_n x = \underbrace{\ln \beta_n}_{\to 0} + \ln x \longrightarrow f(x) \quad \forall x > 0$

$\to 1$

$$\sup_{x>0} |\underbrace{f_n(x) - f(x)}_{= \ln \beta_n \ \forall x}| = |\ln \beta_n| \to 0$$

$\rightsquigarrow \boxed{(f_n) \text{ konvergerar likformigt}}$

• $g(f_n(x)) - g(f(x)) = e^{\ln \beta_n + \ln x} - e^{\ln x} = x(\beta_n - 1)$

$$\max_{x>0} |g(f_n(x)) - g(f(x))| \geq 1 \quad \forall n \quad \rightsquigarrow \boxed{(g_n) \text{ konv. ej likformigt}}$$

$x = \frac{1}{\beta_n - 1}$

Bonus:

Visa att:

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \left(\sum_{n=1}^\infty \frac{1}{1+n^2x^2} \right) \sqrt{x} \, dx = \sum_{n=1}^\infty \frac{1}{n^{3/2}} \int_0^n \frac{\sqrt{x}}{1+x^2} \, dx .$$

Sätt:

$$u_n(x) = \frac{1}{1+n^2x^2} \cdot \sqrt{x} \quad , \quad n \geq 1 .$$

$$\forall \varepsilon > 0 \quad \sup_{x \in [\varepsilon; 1]} |u_n(x)| \leq \frac{1}{1+n^2\varepsilon^2} \rightsquigarrow \sum_{n=1}^\infty \sup_{x\in[\varepsilon,1]} |u_n(x)| < +\infty$$

Weierstraß M-test : $\sum_{n=1}^\infty u_n(x)$ konvergerar likformigt på $[\varepsilon, 1]$

$$\rightsquigarrow \int_\varepsilon^1 \left(\sum_{n=1}^\infty u_n(x) \right) dx$$

$$= \sum_{n=1}^\infty \underbrace{\int_\varepsilon^1 u_n(x)\,dx}_{\int_\varepsilon^1 \frac{\sqrt{x}}{1+n^2x^2}\,dx} = \sum_{n=1}^\infty \beta_n(\varepsilon)$$

$$\int_\varepsilon^1 \frac{\sqrt{x}}{1+n^2x^2} dx = \frac{1}{n^{3/2}} \int_{\varepsilon n}^{n} \frac{\sqrt{x}}{1+x^2} dx \;\not\equiv\; \beta_n(\varepsilon)$$

$$\beta_n(0) = \int_0^n \frac{\sqrt{x}}{1+x^2}\,dx$$

$$|\beta_n(\varepsilon) - \beta_n(0)| = \int_0^{\varepsilon n} \frac{\sqrt{x}}{1+x^2}\,dx \leq \sqrt{\varepsilon n}\,\underbrace{\int_0^\infty \frac{dx}{1+x^2}}_{= C = \pi/2}.$$

$$\Rightarrow \left| \sum_{n=1}^\infty \int_\varepsilon^1 u_n(x)\,dx - \sum_{n=1}^\infty \frac{1}{n^{3/2}} \int_0^n \frac{\sqrt{x}}{1+x^2}\,dx \right|$$

$$\leq \sum_{n=1}^\infty \frac{1}{n^{3/2}} |\beta_n(\varepsilon) - \beta_n(0)| \leq \sqrt{\varepsilon} \left(\sum_{n=1}^\infty \frac{1}{n^2} \right) \cdot C \to 0,$$

då $\varepsilon \to 0^+$

$\boxtimes$