• Lösningsförslag Tenta TMA976, 17 Jan 2020

1. $xy' = y^2 + 3y + 2 = (y+1)(y+2) \rightsquigarrow \dfrac{dy}{(y+1)(y+2)} = \dfrac{dx}{x}$

(*) ledtråd! $\left(\dfrac{1}{y+1} - \dfrac{1}{y+2}\right) dy$

$\rightsquigarrow \ln\left(\dfrac{y+1}{y+2}\right) = \ln x + C$

$\left[\, y(1) = 0 \rightsquigarrow C = \ln \tfrac{1}{2} \,\right] \rightsquigarrow \dfrac{y+1}{y+2} = \dfrac{1}{2}x$

$\rightsquigarrow y + 1 = \frac{1}{2}x(y+2) \rightsquigarrow (1 - \frac{1}{2}x)y = x - 1 \rightsquigarrow \boxed{y(x) = \dfrac{x-1}{1-\frac{1}{2}x}}$ endast giltig för $x \in (-\infty,$

---

2. $y_n'' - 2\alpha_n y_n' + \alpha_n^2 y_n = \underbrace{(y_n' - \alpha_n y_n)'}_{z_n'} - \alpha_n \underbrace{(y_n' - \alpha_n y_n)}_{=: z_n} = x^2 e^{\alpha_n x}$

$z_n(0) = y_n'(0) - \alpha_n y_n(0) = 0$

$\rightsquigarrow z_n(x) = e^{\alpha_n x} \int_0^x \cancel{e^{-\alpha_n t}}\, t^2\, \cancel{e^{\alpha_n t}}\, dt = \frac{1}{3} x^3 e^{\alpha_n x}$

$\rightsquigarrow y_n' - \alpha_n y_n = \frac{1}{3} x^3 e^{\alpha_n x} \rightsquigarrow y_n(x) = e^{\alpha_n x} \int_0^x \cancel{e^{-\alpha_n t}}\, \frac{1}{3} t^3\, \cancel{e^{\alpha_n t}}\, dt = \dfrac{x^4}{12} e^{\alpha_n x}$

$\alpha_n = 4n\ln n$

$\rightsquigarrow y_n\left(\frac{1}{n}\right) = \dfrac{1}{12n^4} e^{\frac{4n \cdot \ln n}{n}} \overset{\forall n \geq 1}{=} \dfrac{1}{12} \to \boxed{\dfrac{1}{12}}$

---

3. $a_n = \dfrac{n^p}{(\ln n)^2} \left| \underbrace{\sin\left(\frac{1}{n}\right) - \ln\left(1 + \frac{1}{n}\right)} \right| \geq 0$

* ledtråd! $\approx \cancel{\frac{1}{n}} - \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right) - \left(\cancel{\frac{1}{n}} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right) = \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$

$b_n = \dfrac{n^{p-2}}{(\ln n)^2} \rightsquigarrow \dfrac{a_n}{b_n} \to \dfrac{1}{2} > 0$

$\rightsquigarrow \left(\sum_{n=5}^{\infty} a_n \text{ konv.} \iff \sum_{n=5}^{\infty} b_n \text{ konv.} \right)$

Jämförelsekriteriet!

Integralkriteriet $\Updownarrow$ $\displaystyle\int_5^{\infty} \frac{x^{p-2}}{(\ln x)^2}\, dx < +\infty$

$\{u = \ln x\}$: $\displaystyle\int_{\ln 5}^{\infty} \frac{e^{(p-1)u}}{u^2}\, du < +\infty$ omm $p \leq 1$.

$\Updownarrow$

$\boxed{p \leq 1}$

---

4. $\cos x(1-x) - 1 + \frac{x}{2}\sin x = \cancel{1} - \frac{1}{2}x^2(1-x)^2 + O(x^4) - \cancel{1} + \frac{x}{2}\left(x - O(x^3)\right)$

$= -\frac{1}{2}x^2(1-x)^2 + \frac{x^2}{2} + O(x^4) = -\frac{1}{2}x^2(1 - 2x + x^2) + \frac{x^2}{2} + O(x^4)$

$= x^3 + O(x^4) \rightsquigarrow \dfrac{\cos x(1-x) - 1 + \frac{x}{2}\sin x}{x^3} = 1 + O(x) \to \boxed{1}$ då $x \to 0$

---

5. $\displaystyle\sum_{n=1}^{\infty} a_n x^n$, $a_n = \begin{cases} 0 & n \neq k^2 \text{ för ngt. } k \\ \frac{(-1)^k}{k} & n = k^2 \text{ för ngt. } k \end{cases}$

$\rightsquigarrow |a_n|^{1/n} = \begin{cases} 0 & n \neq k^2 \quad \to 0 \\ 1/k^{1/k^2} & n = k^2 \quad \to 1 \end{cases}$

Rotkriteriet: $\rightsquigarrow$ Konv. (absolut) om $|x| < 1$

Div. om $|x| > 1$

$\underline{x = 1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ betingat konvergent

$\underline{x = -1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{n^2}}{n}$ divergerar!

$(-1)^n(-1)^{n^2} = \begin{cases} 1 & n \text{ jämn} \\ 1 & n \text{ udda } (\iff n^2 \text{ udda}) \end{cases} = 1 \ \forall n$

6. Fixera $0 < \varepsilon < 1$, sätt $u_n(x) = x^n (1-x)^{1/3}$

$$\rho_n := \max_{0 \le x \le 1-\varepsilon} |u_n(x)| \le (1-\varepsilon)^{n+1/3} \rightsquigarrow \sum_n \rho_n < \infty$$

$\rightsquigarrow$ Weierstraß M-test: $\sum_{n=0}^{\infty} u_n(x)$ konv. likformigt på $[0, 1-\varepsilon]$

$$\rightsquigarrow \sum_{n=0}^{\infty} \int_0^{1-\varepsilon} u_n(x)\,dx = \int_0^{1-\varepsilon} \underbrace{\left(\sum_{n=0}^{\infty} x^n\right)}_{=\frac{1}{1-x}\ \text{(geom. summa)}} (1-x)^{1/3}\,dx = \int_0^{1-\varepsilon} (1-x)^{-2/3}\,dx$$

$$= \left[-3(1-x)^{1/3}\right]_0^{1-\varepsilon} = 3(1-\varepsilon^{1/3})$$

$$\to \boxed{3, \text{ då } \varepsilon \to 0.}$$

---

7. $$\frac{y \sin xy + x \sin x^2}{x^2+y^2} = \left\{ \begin{array}{l} \text{Antag att} \\ x \neq 0, y \neq 0 \end{array} \right\} = \frac{1}{x^2+y^2}\left(y^2 \frac{\sin xy}{xy} + x^2 \frac{\sin x^2}{x^2} \right)$$

$$= \frac{1}{x^2+y^2}\left(y^2\left(\frac{\sin xy}{xy} - 1\right) + x^2\left(\frac{\sin x^2}{x^2} - 1\right) + x^2 + y^2 \right)$$

$$= 1 + \underbrace{\underbrace{\frac{y^2}{x^2+y^2}}_{\le 1} \underbrace{\left(\frac{\sin xy}{xy} - 1\right)}_{\to 0} + \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} \underbrace{\left(\frac{\sin x^2}{x^2} - 1\right)}_{\to 0}}_{\to 0} \to \boxed{1}$$

Kolla även $y=0$
