Exam for the course “Options and Mathematics” (CTH[*MVE095*], GU[*MMG810*]) 2019/20

Telefonvakt/Rond: ??

This is a sample exam!

REMARKS: (1) No aids permitted (2) Minor errors in the calculations will be forgiven, but remember that fractions look nicer when you simplify them!

Part I

1. Assume that the market is frictionless, arbitrage free and that the assets pay no dividend. Prove that the value of call options is a non-increasing and convex function of the strike price (max. 2 points). **Solution.** See lecture notes, theorem 1.3.

2. Derive the Black-Scholes price of European call options (max. 2 points). **Solution.** See lecture notes, theorem 6.6.

3. Give and explain the definition of optimal exercise time of American put options (max. 2 points). **Solution.** See lecture notes, definition 1.2. The interpretation of optimal exercise time is that of a time at which exercising the American put is as lucrative as selling it, hence the buyer takes full advantage of the derivative value by exercising it at this time.

4. Decide whether the following statements are true or false an explain your answer (max. 2 points):

   (a) In a frictionless, arbitrage-free market with positive risk-free rate the value of European put options is non-decreasing with maturity.

   (b) In a frictionless, arbitrage-free market with positive risk-free rate the value of American put options is non-decreasing with maturity.

   Solution. (a) is false, e.g., because the value of European put options is bounded above by $Ke^{-r(T-t)}$ and thus it tends to zero as $T \to \infty$ when $r > 0$. This has a simple intuitive explanation: since the maximum pay-off for a European put is K and it is payed at maturity, then receiving the pay-off in the future becomes less and less valuable the longer one has to wait for it. (b) is true. The difference with (a) is the option of earlier exercise. Let $T_2 > T_1$ and $\widehat{P}_i$ be the value at time 0 of an American put expiring at time T_i, $i = 1, 2$ and with strike K. The claim (b) is that $\widehat{P}_2 \geq \widehat{P}_1$. Suppose this is not true, i.e., $\widehat{P}_2 < \widehat{P}_1$. At time $t = 0$ we buy $\widehat{P}_2$ and sell $\widehat{P}_1$ and invest the remaining quantity $\widehat{P}_1 - \widehat{P}_2$ in the money market. The resulting portfolio is an arbitrage in the interval $[0, T_1]$. In fact, it has zero value and if the owner of $\widehat{P}_1$ exercises then we just exercise our own $\widehat{P}_2$ to pay him/her off (the pay-off is the same because the two put options have the same strike). Thus the value of the portfolio at time T_1 is at least equal to the value of the risk-free asset, which is positive, hence the portfolio is an arbitrage.

Part II

1. A European derivative on a stock pays the amount

$$Y = (\min(S(T) - 10, 20 - S(T), 2))_+ - 1.$$

Draw the pay-off function of the derivative and find a constant portfolio of European call/put options that replicates the derivative (max 4 points).

Solution. Let $Y = Q - 1$, where $Q = (\min(S(T) - 10, 20 - S(T), 2))_+$. Draw the graph of the straight lines $y = x - 10$, $y = 20 - x$ and $y = 2$ and take their minimum to find the graph of Q. Shift the graph of Q one unit below on the y-axis to find the graph of Y, see figure below. A replicating portfolio is a portfolio $\mathcal{A}$ such that $\Pi_Y(t) = V_{\mathcal{A}}(t)$. By Theorem 1.1 (b) (with $\mathcal{C}_A = 0$) it suffices to find the portfolio $\mathcal{A}$ such that $Y = V_{\mathcal{A}}(T)$. Thus to solve the second part of the exercise we have to write Y as a linear combination of pay-offs of calls an puts. There are several ways to do this. From one hand one can see that

$$Y = -(11 - S(T))_+ + (10 - S(T))_+ + (S(T) - 11)_+ - (S(T) - 12)_+ - (S(T) - 18)_+ + (S(T) - 20)_+$$

hence the derivative is replicated by a portfolio that consists of -1 share of the put with strike 11, 1 share of the put with strike 10, 1 share of the call with strike 11, 1 share of the call with strike 12, 1 share of the call with strike 18 and 1 share of the call with strike 20. On the other hand one can see that

$$Y = Q - 1 = (S(T) - 10)_+ - (S(T) - 12)_+ - (S(T) - 18)_+ + (S(T) - 20)_+ - 1$$

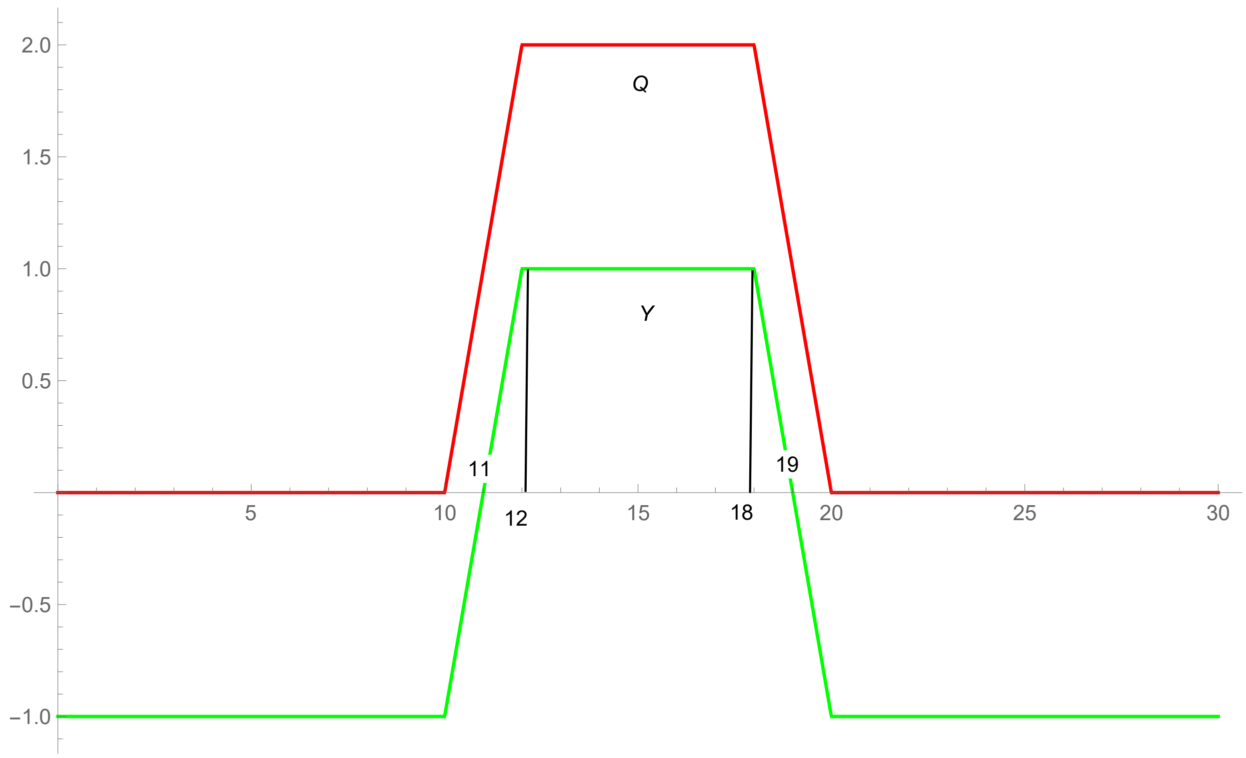
Now write, for all $K > 0$,

$$1 = (S(T) - K) - (S(T) - (K+1)) = (S(T) - K)_+ - (K - S(T))_+ - (S(T) - (K+1))_+ + (K + 1 - S(T))_+,$$

where we used that $(x - K) = (x - K)_+ - (K - x)_+$. Hence for all $K > 0$ there holds

$$\begin{aligned} Y &= (S(T) - 10)_+ - (S(T) - 12)_+ - (S(T) - 18)_+ + (S(T) - 20)_+ \\ &\quad - (S(T) - K)_+ + (K - S(T))_+ + (S(T) - (K + 1))_+ - (K + 1 - S(T))_+ \end{aligned}$$

For $K = 10$ we recover the previous replicating portfolio.



2. Consider a binomial market with parameters $e^u = \frac{7}{4}$, $e^d = \frac{1}{2}$, $S(0) = 1$, $p = 3/4$, $e^r = 9/8$.

   a) Compute the binomial price at $t = 0, 1, 2$ of an American put with strike $K = 3/4$ and maturity $T = 2$ (max. 1 point).

   b) Compute the binomial price at $t = 0, 1, 2$ of a European call with strike $K = 3/4$ and maturity $T = 2$ (max. 1 point).

   c) A derivative $\mathcal{U}$ gives to its owner the right to convert $\mathcal{U}$ at time $t = 1$ into either the European call or the American put defined above. Compute the binomial price of $\mathcal{U}$ at time $t = 0$ (max. 1 point).

   d) Describe the strategy that maximizes the expected return for the holder of $\mathcal{U}$ (max. 1 point).

Solution. The binomial tree for the stock price is

$$
\begin{tikzcd}
 & & S(2)=\frac{49}{16} \\
 & S(1)=\frac{7}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
S_0=1 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=\frac{7}{8} \\
 & S(1)=\frac{1}{2} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & S(2)=\frac{1}{4}
\end{tikzcd}
$$

The martingale probability is

$$q_u = \frac{e^r - e^d}{e^u - e^d} = 1/2 = q_d$$

We compute the price $\widehat{P}(t)$ of the American put by the recurrence formula

$$\widehat{P}(2) = (3/4 - S(2))_+, \quad \widehat{P}(t) = \max[3/4 - S(t), e^{-r}(q_u \widehat{P}^u(t+1) + q_d \widehat{P}^d(t+1))],\ t = 0, 1,$$

and the price $C(t)$ of the European call by the recurrence formula

$$C(2) = (S(2) - 3/4)_+, \quad C(t) = e^{-r}(q_u C^u(t+1) + q_d C^d(t+1)),\ t = 0, 1,$$

So doing we obtain

$$
\begin{tikzcd}
 & & \widehat{P}(2)=0 \\
 & \widehat{P}(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
\widehat{P}(0)=\frac{1}{9} \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{P}(2)=0 \\
 & \widehat{P}(1)=\frac{1}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \widehat{P}(2)=\frac{1}{2}
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & C(2) = \frac{37}{16} \\
 & C(1) = \frac{13}{12} \arrow[ur, "u"] \arrow[dr, "d"] & \\
C(0) = \frac{41}{81} \arrow[ur, "u"] \arrow[dr, "d"] & & C(2) = \frac{1}{8} \\
 & C(1) = \frac{1}{18} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & C(2) = 0
\end{tikzcd}$$

This concludes part (a)-(b) of the exercise. For part (c) we use that the pay-off of $\mathcal{U}$ is

$$Y = \max(\widehat{P}(1), C(1)) = \begin{cases} 13/12 & \text{if } S(1) = 7/4 \\ 1/4 & \text{if } S(1) = 1/2 \end{cases}$$

As Y depends only on $S(1)$, then we can treat $\mathcal{U}$ as a European derivative on the stock (namely, a **second derivative**) and therefore we can compute its price at time $t = 0$ using the binomial model, that is

$$\Pi_Y(0) = e^{-r}(q_u \Pi_Y^u(1) + q_d \Pi_Y^d(1)) = \frac{8}{9}(\frac{1}{2} \cdot \frac{13}{12} + \frac{1}{2} \cdot \frac{1}{4}) = \frac{16}{27}$$

This concludes part (c). As to part (d), there are 5 possibilities. (i) The stock price goes up at time 1, the investor chooses the call and waits until maturity. In this case the expected return is $\frac{37}{16}p + \frac{1}{8}(1-p) - \frac{16}{27} = \frac{35}{16}p - \frac{101}{216}$. (ii) The stock price goes up at time 1 and the investor chooses the put. This is of course a dummy decision, because the put is worthless; the return would be $-16/27 < 0$. (iii) The stock price goes down at time 1, the investor chooses the call and waits until maturity. In this case the expected return is $\frac{1}{8}p - \frac{16}{27} < 0$. (iv) The investor chooses the put and exercise it immediately. In this case the return for the owner of $\mathcal{U}$ is $1/4 - 16/27 < 0$. As $1/4 > p/8$, this strategy entails a lower loss than (iii). (v) The stock price goes down at time 1, the investor chooses the put and waits until maturity. In this case the expected return $\frac{1}{2}(1-p) - \frac{16}{27} = -\frac{5}{54} - p < 0$. For $p = 3/4$ the choice (v) entails a larger expected loss than (iv). Thus we see that in order to maximize the expected return the owner of the derivative should buy the call if the stock price goes up at time 1, buy the put and exercise immediately if the stock price goes down at time 1. In the first case the expected return is positive, in the second case it is negative (i.e., the investor incurs in a loss).

Remark 1. Note that the expected returns have been computed based on the information available at time $t = 1$, because the exercise asks to derive the strategy that the investor should follow, i.e., the decision that the investor should take at time 1 (which is the only time at which the investor can change the portfolio). Note also that the strategy to follow depends on the value of p. For $p < 1/2$, one should not exercise the American put at time 1 if one wants to minimize the expected loss.

Remark 2. One might have a different interpretation of what it means that a strategy maximizes the expected return. For instance, in the solution given above the possibility that the investor may sell the call (or put) at time $t = 1$ was not contemplated. A different answer to (d) than the one proposed could still be considered correct, and thus awards 1 point, *if it is well-motivated*.

3. The European physically settled digital put option is the derivative with pay-off $Y = S(T)H(K - S(T))$ at maturity T, where $S(t)$ is the price of the underlying stock, K is the strike price of the option and $H(z)$ is the Heaviside function. Derive the Black-Scholes price of this derivative and the number of stock shares in the hedging portfolio (max 4 points).

Solution. To compute the Black-Scholes price we use the formula

$$\Pi_Y(t) = v(t, S(t)), \quad v(t, x) = e^{-r\tau} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}\right) e^{-\frac{1}{2}y^2}\,\frac{dy}{\sqrt{2\pi}},$$

where $\tau = T - t$ and $g(z)$ is the pay-off function of the derivative, see equation (6.12) in the lecture notes. Using $g(z) = zH(K - z)$ we find

$$v(t, x) = e^{-r\tau} \int_{\mathbb{R}} xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y} H\left(K - xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}\right) e^{-\frac{1}{2}y^2}\,\frac{dy}{\sqrt{2\pi}}$$

The Heaviside function equals 1 when the argument is positive and zero otherwise. Since

$$K - xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y} > 0 \ \text{ if and only if } y < -d_2, \quad d_2 = \frac{\log\frac{x}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

then

$$\begin{aligned} v(t, x) &= e^{-r\tau} \int_{-\infty}^{-d_2} xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y-\frac{1}{2}y^2}\,\frac{dy}{\sqrt{2\pi}} = x\int_{-\infty}^{-d_2} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\,\frac{dy}{\sqrt{2\pi}} \\ &= x\int_{-\infty}^{-d_2-\sigma\sqrt{\tau}} e^{-\frac{1}{2}z^2}\,\frac{dz}{\sqrt{2\pi}} = x\Phi(-d_1), \quad d_1 = d_2 + \sigma\sqrt{\tau}. \end{aligned}$$

The number of stock shares in the hedging portfolio is $h_S(t) = \partial_x v(t, S(t))$, where

$$\begin{aligned} \partial_x v(t, x) &= \partial_x[x\Phi(-d_1)] = \Phi(-d_1) + x\phi(-d_1)\partial_x(-d_1) = \Phi(-d_1) - x\phi(-d_1)\partial_x(d_2) \\ &= \Phi(-d_1) - \frac{\phi(-d_1)}{\sigma\sqrt{\tau}}. \end{aligned}$$

where $\phi(z) = \Phi'(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ is the standard normal density.