MVE550 2019 Lecture 4

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Some words you need to learn about Markov chains

MARKOV CHAIN, STATE SPACE, TIME-HOMOGENEOUS, TRANSITION MATRIX, STOCHASTIC MATRIX, LIMITING DISTRIBUTION, STATIONARY DISTRIBUTION, POSITIVE MATRIX, REGULAR TRANSITION MATRIX, TRANSITION GRAPH, RANDOM WALK, WEIGHTED GRAPH, **ACCESSIBLE STATES, COMMUNICATING STATES, EQUIVALENCE RELATION, COMMUNICATION CLASSES, IRREDUCIBILITY, RECURRENT STATES, TRANSIENT STATES, CLOSED COMMUNICATION CLASSES, CANONICAL DECOMPOSITION, IRREDUCIBLE MARKOV CHAINS, POSITIVE RECURRENT STATES, NULL RECURRENT STATES, PERIODICITY, APERIODIC, ERGODIC MARKOV CHAINS, TIME REVERSIBILITY, DETAILED BALANCE CONDITION, ABSORBING STATES, ABSORBING MARKOV CHAINS, FUNDAMENTAL MATRIX, ...**

Overview lecture 3

- Recurrent and transient states; communication classes.
- The limit theorem for finite irreducible Markov chains.
- Periodicity
- The limit theorem for ergodic Markov chains.
- Absorbing chains
- Time reversibility

Moving between states

- State j is *accessible* from state i if $(P^n)_{ij} > 0$ for some $n \geq 0$.
- States i and j *communicate* if i is accessible from j and j is accessible from i.
- "Communication" is *transitive*, i.e., if i communicates with j and j communicates with k, then i communicates with k.
- Communication is an *equivalence relation*, subdividing all states int *communication classes*.
- Communication classes can be found for example by drawing transition graphs.
- A Markov chain is *irreducible* if it has exactly one communication class.

Recurrence and transience

- Let T_j be the *first passage time* to state j: $T_j = \min\{n > 0 : X_n = j\}$.
- Define f_j as the probability that a chain starting at j will return to j:

$$f_j = P(T_j < \infty \mid X_0 = j)$$

- A state j is *recurrent* if a chain starting at j will eventually revisit j, i.e., if $f_j = 1$.
- A state j is *transient* if a chain starting at j has a positive probability of never revisiting j, i.e., if $f_j < 1$.
- Note: The expected number of visits at j when the chain starts at i is given by $\sum_{n=0}^{\infty} (P^n)_{ij}$.
- j is recurrent if and only if $\sum_{n=0}^{\infty} (P^n)_{jj} = \infty$.
- j is transient if and only if $\sum_{n=0}^{\infty} (P^n)_{jj} < \infty$.

Communication classes

- The states of a communication class are either all recurrent or all transient.
- The states of a finite irreducible Markov chain are all recurrent.
- Note: There are infinite irreducible Markov chains where all states are transient.
- Example: Simple random walk with non-symmetric probabilities.
- If a state is recurrent, only states inside its communication class are accessible from it.
- If no states outside a finite communication class are accesible from it, then the class consists of recurrent states.

Finite irreducible Markov chains

- Recall: In a finite irreducible Markov chain, all states are recurrent.
- **Limit Theorem for Finite Irreducible Markov Chains**: Let $\mu_j = E(T_j \mid X_0 = j)$ be the expected return time to j. Then $\mu_j < \infty$ and the vector v with $v_i = 1/\mu_i$ is a stationary distribution. Furthermore,
$$v_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} (P^m)_{ij}.$$
- NOTE: All finite regular Markov chains are finite irreducible Markov chains.
- NOTE: The conclusion is *weaker* than that for finite regular Markov chains.
- Example: The theorem holds for the chain with transition matrix
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Extention to infinite irreducible Markov chains

- In a finite irreducible Markov chain, all states are recurrent, and all expected return times μ_j are finite.
- In a Markov chain, states may be recurrent but with infinite expected return times. Such states are called *null recurrent*, while recurrent states with finite expected return times are called *positive recurrent*.
- The previous theorem may be extended to infinite irreducible Markov chains where all states are positive recurrent.

Periodicity

- The *period* of a state i is the greatest common divisor of all $n > 0$ such that $(P^n)_{ii} > 0$.
- Show: All states of a communication class have the same period.
- A Markov chain is *periodic* if it is irreducible and all states have period greater than 1.
- A Markov chain is *aperiodic* if it is irreducible and all staes have period equal to 1.

Ergodic Markov chains

- A Markov chain is *ergodic* if
  - it is irreducible
  - it is aperiodic
  - all states are positive recurrent (i.e., have finite expected return times). (Always happens if the state space is finite).
- **Fundamental Limit Theorem for Ergodic Markov Chains**: There exists a unique positive stationary distribution v which is the limiting distribution of the chain.
- Note: We can also show that a finite Markov chain is ergodic if and only if it its transition matrix is regular.

Absorbing chans

- State i is *absorbing* if $P_{ii} = 1$.
- A Markov chain is *absorbing* if it has at least one absorbing state.
- By reordering the states, the transition matrix for an absorbing chain can be written in block form

$$P = \begin{bmatrix} Q & R \\ \mathbf{0} & I \end{bmatrix}.$$

  where I is the identity matrix, $\mathbf{0}$ is a matrix of zeros, and Q corresponds to transient states.
- We can prove by induction that

$$P^n = \begin{bmatrix} Q^n & \left(I + Q + Q^2 + \cdots + Q^{n-1}\right) R \\ \mathbf{0} & I \end{bmatrix}.$$

- Taking the limit and using $\lim_{n\to\infty} Q^n = 0$ we get

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \mathbf{0} & (I - Q)^{-1} R \\ \mathbf{0} & I \end{bmatrix}.$$

- $F = (I - Q)^{-1} = \lim_{n\to\infty} I + Q + \cdots + Q^n$ is called the *fundamental matrix*.

Absorbing chains, cont

- The probability to be absorbed in a particular absorbing state given a start in a transient state is given by the entries of FR.
- Further, the expected number of visits in state j for a chain that starts in the transient state i is given by F_{ij}. (Show this).
- Thus, the expected number of steps until absorbtion is given by the vector $F\mathbf{1}^t$.
- Note: Given an irreducible Markov chain. To compute the expected number of steps needed to go from state i to the first visit to state j, one can change the chain into one where state j is absorbing, and compute the expected number of steps until absorbtion using the theory above.

Example: First detection of a particular sequence

- Assume you want to find the expected number of steps until you detect HTTH in a sequence of fair coin flips.
- Build a Markov chain where the states indicate how far into the sequence you have read so far. Make the state HTTH absorbing.
- Find the transition matrix in canonical block form.

Time reversibility

Let P be the transition matrix of an irreducible Markov chain with stationary distribution v.

- The chain is "time reversible" if, after reaching its stationary distribution, it looks the same moving foreard as backwards, i.e., $\pi(X_k = i, X_{k+1} = j) = \pi(X_{k+1} = i, X_k = j)$.
- This translates to $v_i P_{ij} = v_j P_{ji}$ for all i, j: The *detailed balance condition*.
- Note: If x is a probability vector satisfying $x_i P_{ij} = x_j P_{ji}$ for all i, j, then necessarily x is the stationary distribution, so that $x = v$.
- Show: If a Markov chain is defined as a random walk on a weighted undirected graph, then it is time reversible.
- Show: If a finite Markov chain is time reversible, it can be represented as a random walk on a weighted undirected graph.