MVE550 2018 Lecture 6

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Overview

- Branching processes.
- Mean and variance of generation size.
- Probability generating functions.
- Theory about the probability of extinction. Theorem and examples.

Branching processes

A branching process is discrete Markov chain $Z_0, Z_1, \ldots, Z_n, \ldots$ where

- the state space is the non-negative integers
- $Z_0 = 1$
- 0 is an absorbing state
- Z_n is the sum $X_1 + X_2 + \cdots + X_{Z_{n-1}}$, where the X_j are independent random non-negative integers with the *offspring distribution*
- Connecting each of the Z_n individuals in generation n with their offspring in generation $n+1$ we get a tree illustrating the branching process
- To avoid trivial cases, we assume $a_0 > 0$ and $a_0 + a_1 < 1$, where $a_i = \Pr(X_j = i)$.

Mean and variance of generation size

Let μ be the expectation of the offspring distribution. Then

- We get $\mathrm{E}(Z_n) = \mu^n$
- We say that
  - The process is *subcritical* if $\mu < 1$
  - The process is *critical* if $\mu = 1$
  - The process is *supercritical* if $\mu > 1$
- We get for the variance: If $\mu = 1$:

$$\mathrm{Var}(Z_n) = n\sigma^2$$

and if $\mu \neq 1$

$$\mathrm{Var}(Z_n) = \sigma^2 \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1}$$

where σ^2 is the variance of the offspring distribution.

Probability generating functions

- For *any* discrete random variable X taking values in $\{0, 1, 2, \ldots, \}$ define the probability generating function $G(s)$, or $G_X(s)$, as

$$G(s) = \mathrm{E}\left(s^X\right)$$

- Two such discrete random variables that have the same probability generating function must have the same distribution
- We have
  - $G(1) = 1$
  - $G^{(j)}(0) = j! \Pr(X = j)$
  - If X and Y are independent, $G_{X+Y}(s) = G_X(s)G_Y(s)$
  - $\mathrm{E}(X) = G'(1)$
  - $\mathrm{Var}(X) = G''(1) + G'(1) - G'(1)^2$

Extinction probability theorem

THEOREM

- Let G be the probability generating function for the offspring distribution for a branching process. The probability of eventual extinction is the smallest positive root of the equation $s = G(s)$.
- Also: In the subcritical and critical cases, the extinction probability is 1.