# MVE550 2019 Lecture 10

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# Where are we?

- In the beginning of the course, we defined a stochastic process as a collection $\{X_t, t \in I\}$ of random variables with a common state space $S$.
- So far, the set $I$ has been the non-negative integers. We now move on to processes where $I$ is a non-countable set, for example all positive real numbers, or all subsets of $\mathbb{R}^2$.
- Chapters 6 and 7 of Dobrow concern such stochastic processes where the state space $S$ is discrete.
- In Chapter 8 of Dobrow we look at the situation when the random variables $X_t$ are continuous variables.

# Overview

- Three equivalent definitions of a Poisson process.
- A number of important and useful properties.
- Examples and example computations!
- Spatial Poisson processes and inhomogeneous Poisson processes.

# Counting processes

- A *counting process* $\{N_t, t \in I\}$ is a stochastic process where $I = \mathbb{R}_0^+$, where the state space is the non-negative integers, and where $0 \leq s \leq t$ implies $N_s \leq N_t$.
- Informally, when $s < t$, $N_t - N_s$ counts the number of "events" in $(s, t]$.
- $N_t$ is a function of $t$ that is a right-continuous step function.

# Poisson process: Definiton 1

- A Poisson process $\{N_t\}_{t \geq 0}$ with parameter $\lambda > 0$ is a counting process fulfilling
  - $N_0 = 0$.
  - $N_t \sim \text{Poisson}(\lambda t)$ for all $t > 0$.
  - *Stationary increments:* $N_{t+s} - N_s$ has the same distribution as $N_t$.
  - *Independent increments:* $N_t - N_s$ and $N_r - N_q$ are independent, when $0 \leq q < r \leq s < t$.
- Note: Not obvious that such a process exists.
- Note: $\text{E}(N_t) = \lambda t$. Thus what one is counting occurs with a *rate* of $\lambda$ items per time unit.

# Poisson process: Definition 2

- Let $X_1, X_2, \ldots,$ be a sequence of iid exponential random variables with parmeter $\lambda$. Define $N_0 = 0$ and, for $t > 0$,

$$N_t = \max\{n : X_1 + \cdots + X_n \le t\}.$$

  Then $\{N_t\}_{t \ge 0}$ is a Poisson process with parameter $\lambda$.
- We call $S_n = X_1 + \cdots + X_n$ the *arrival times* of the process.
- We call $X_k = S_k - S_{k-1}$ the *inter-arrival times* of the process.
- This provides an easy way to simulate a Poisson process.

# Memorylessness of the exponential distribution

- A random variable $X$ is called *memoryless* if

$$P(X > s + t \mid X > s) = P(X > t)$$

  for all $s > 0$, $t > 0$.
- The exponential distribution is memoryless, and is the only memoryless continuous random variable.
- Consider the consequences of this when using the exponential as a model.

# Minimum and sum of independent exponentially distributed variables

- Define $M = \min(X_1, \ldots, X_n)$ where, independently for each $i$, $X_i \sim \text{Exponential}(\lambda_i)$. Then:
  - $M \sim \text{Exponential}(\lambda_1 + \cdots + \lambda_n)$.
  - $P(M = X_k) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}$.
- Let $S_n = X_1 + \cdots + X_n$ where, independently for each $i$, $X_i \sim \text{Exponential}(\lambda)$. Then $S_n \sim \text{Gamma}(n, \lambda)$.

# Poisson process: Definition 3

- Introduce/review the $o(h)$ and $o(g(h))$ notation.
- A Poisson process $\{N_t\}_{t\geq 0}$ with parameter $\lambda$ is a couting process fulfilling
  - $N_0 = 0$.
  - The process has stationary and independent increments.
  - We have

$$\begin{aligned} P(N_h = 0) &= 1 - \lambda h + o(h) \\ P(N_h = 1) &= \lambda h + o(h) \\ P(N_h > 1) &= o(h) \end{aligned}$$

- All the three definitions of a Poisson process are equivalent.

# Thinned poisson processes

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with parameter $\lambda$. Assume each arrivial is "marked" as "type $k$", for one of $n$ types, with probability $p_k$, where $p_1 + \cdots + p_n$. Let $N_t^{(k)}$ be the count of the number of arrivals of type $k$ by time $t$. Then

- $\left\{N_t^{(k)}\right\}_{t\geq 0}$ is a Poisson process with parameter $p_k\lambda$.
- The processes
$$\left\{N_t^{(1)}\right\}_{t\geq 0}, \ldots, \left\{N_t^{(n)}\right\}_{t\geq 0}$$
are independent.

# Superposition process

Assume

$$\left\{N_t^{(1)}\right\}_{t\geq 0}, \ldots, \left\{N_t^{(n)}\right\}_{t\geq 0}$$

are independent Poisson processes with parameters $\lambda_1, \ldots, \lambda_n$, respectively. Define, for $t > 0$,

$$N_t = N_t^{(1)} + \cdots + N_t^{(n)}.$$

Then $\{N_t\}_{t\geq 0}$ is a Poisson process with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$.

# Uniform distribution when count is fixed

Let $S_1, S_2, \ldots,$ be the arrival times of a Poisson process with parameter $\lambda$. **Conditional on** $N_t = n$, we have

- The joint density function for $S_1, \ldots, S_n$ is uniform on the set $0 < s_1 < s_2 < \ldots, < s_n < t$.
- Equivalently, if $U_1, \ldots, U_n$ are iid uniform on $[0, t]$, and if $U_{(1)}, \ldots, U_{(n)}$ is the ordering of these random variables, then $(S_1, \ldots, S_n)$ and $(U_{(1)}, \ldots, U_{(n)})$ have the same distribution.
- The upshot: If we want to simulate a Poisson process on an interval $[0, t]$, we may first simulate $N_t$ (the total number of "events") and then independently simulate the arrival times of each of the $N_t$ events uniformly on $[0, t]$.

# Spatial Poisson processes

- A collection of random variables $\{N_A\}_{A \subseteq \mathbb{R}^d}$ is a spatial Poisson process with parameter $\lambda$ if
  - For each bounded set $A \subseteq \mathbb{R}^d$, $N_A$ has a Poisson distribution with parameter $\lambda|A|$.
  - Whenever $A \subseteq B$, $N_A \leq N_B$.
  - Whenever $A$ and $B$ are disjoint sets, $N_A$ and $N_B$ are independent.
- How to simulate
- One may use simulations to estimate properties such as the average distance to the nearest neighbour (or the third nearest neighbour or whatever).
- Very useful model in practice.

# Non-homogeneous Poisson processes

- A counting process $\{N_t\}_{t \geq 0}$ is a *non-homogeneous* Poisson process with intensity function $\lambda(t)$ if
  - $N_0 = 0$.
  - For $t > 0$,
    $$N_t \sim \text{Poisson}\left(\int_0^t \lambda(x)\,dx\right)$$
  - It has independent increments.
- Again a very flexible and useful model in practice.
- One may have non-homogeneous spatial Poisson processes.