Stochastic Processes Solutions

November 2019

- (1.6) A: $2/(n+1)$
- (1.9a) A: 1/2
  (1.9b) A: 13/48
  (1.9c) A: 6/13
  (1.9d) A: 1/4
  (1.9e) A: 1/8
- (1.10) A: 5/11
- (1.14a) A: $f_{X|Y}(x|y) = 2e^{-(x-y)}, x > y$
  (1.14b) A: The distribution is uniform on (0,x).
- (1.16) A: 21/17
- (1.20a) A: 14
  (1.20b) A: $2^{k+1} - 2$
- (2.2a) A: 1/3
  (2.2b) A: 5/36
  (2.2c) A: 1/4
  (2.2d) A: 0
- (2.7) A: Transition matrix is P^3
- (2.10) A; See Figure 1.
- (2.14a) A:

$$P_{ij} = \begin{cases} i/n, & \text{if } \; j = i \\ (n-i)/n & \text{if } \; j = i+1 \end{cases} \tag{1}$$

  (2.14b) A: 195/512
- (3.9a) A: Yes
  (3.9b) A: No, take

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

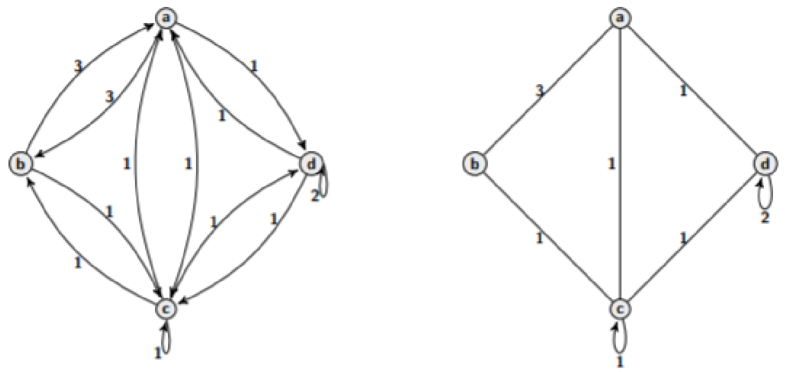

Figure 1: Exercise 2.12

- (3.14a) A: Long term probability of an episode day is 0.489
  (3.14b) A: 178.62
  (3.14c) A: 2.04
- (3.18) A: Let $f_x = \mathbb{E}[T_b|X_0 = x]$ for $x = a, b, c$. Then $f_a = f_c = 2, f_b = 3$
- (3.38a) A:

$$P_{ij} = \binom{5-i}{j-i}\left(\frac{1}{6}\right)^{j-i}\left(\frac{5}{6}\right)^{5-j} \tag{3}$$

  (3.38b) A: 13.0237
- (3.41) A $\pi = (3/13, 5/26, 3/13, 9/26)$
- (4.5a) A: The kth factorial moment is $G^{(k)}(1)$
  (4.5b) A: $n!p^k/((n-k)!)$
- (4.6a) A: $G_Z(s) = e^{-\lambda p(1-s)}$
  (4.6b) The generating function is that of a Poisson random variable with parameter λp, which gives the distribution of Z.
- (4.12a) A: $\mu = 5/4$
  (4.12b) A: $G(s) = 1/4 + s/4 + s^2/2$
  (4.12c) A: 1/2
  (4.12d) A: $G_2(s) = 1/32(11 + 4s + 9s^2 + 4s^4 + 4s^4)$
  (4.12e) A: 11/32
- (4.20a) A: e=1
  (4.20b) A:

$$G_n(s) = \frac{n-(n-1)s}{n+1-ns} \tag{4}$$

(4.20c) A: $P(T = n) = 1/(n(n + 1))$

- (4.26a) A: Expected winnings: -0.708
  (4.26b) A:

$$
\begin{aligned}
P(T = 1) &= 0.9118 && (5)\\
P(T = 2) &= 0.0659 && (6)\\
P(T = 3) &= 0.0161 && (7)\\
P(T = 4) &= 0.0044 && (8)
\end{aligned}
$$

- (5.2) Long term average is $1/k$
- (5.4c) With numerical software one can get

$$
\begin{aligned}
\mu_{10} &= 2.91667 && (9)\\
\mu_{100} &= 27.7921 && (10)\\
\mu_{1000} &= 276.546 && (11)
\end{aligned}
$$

- (5.4d) $(5 - \sqrt{5})/10$.
- (5.8a)

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

- (5.8b)

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/2 & 7/18 & 2/18 & 0 \\ 0 & 0 & 1/2 & 5/12 & 1/12 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{13}$$

- (5.8c) One can check that the stationary distribution is the binomial distribution.
- (6.2a) $\mathbb{E}[N_3 N_4] = 54$
- (6.2b) $\mathbb{E}[X_3 X_4] = 1/4$
- (6.2c) $\mathbb{E}[S_3 S_4] = 15/4$
- (6.5) The error is that only the distribution of N_3 is equal to that of $N_6 - N_3$, the random variables are not equal.

- (6.8a) 0.00679
- (6.8b) 0.435
- (6.8c) 0.0656
- (6.8d) 2/5
- (6.16a) 0.1255
- (6.16b) 0.2485
- (6.16c) 0.0376
- (6.24a) $\lambda/(\lambda+\mu)$
- (6.24b) $(\lambda+\mu)e^{-(\lambda+\mu)}$
- (6.24c) $4\lambda^2\mu e^{-2(\mu+\lambda)}$
- (6.32) $\lambda/(\lambda+\mu)$
- (7.1)

  $$\tilde{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \tag{14}$$

  $q_a = 1, q_b = 2, q_c = 4$
- (7.5)

  $$Q = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 3 & -3 \end{bmatrix} \tag{15}$$

- (7.14a)

  $$Q = \begin{bmatrix} -4 & 4 & 0 \\ 1 & -7 & 6 \\ 6 & 2 & -8 \end{bmatrix} \tag{16}$$

- (7.14b) $\pi = (11/25, 8/25, 6/25)$
- (7.14c)

  $$\tilde{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1/7 & 0 & 6/7 \\ 6/8 & 2/8 & 0 \end{bmatrix} \tag{17}$$

- (7.14d) $\psi = (11/37, 14/37, 12/37)$

- (7.20) Mean number of customers is λ/μ
- (7.23a)

$$Q = \begin{bmatrix} -1/2 & 1/2 & 0 & 0 & 0 \\ 1/10 & -3/5 & 1/2 & 0 & 0 \\ 0 & 1/5 & -7/10 & 1/2 & 0 \\ 0 & 0 & 3/10 & -11/20 & 1/4 \\ 0 & 0 & 0 & 2/5 & -2/5 \end{bmatrix} \tag{18}$$

- (7.23b) $\pi = (0.0191, 0.0955, 0.2388, 0.3979, 0.2487)$. Long term average is 2.76
- (7.23c) 0.188
- (7.26)

$$Q = \begin{bmatrix} -3/6 & 3/6 & 0 & 0 \\ 1/24 & -9/24 & 8/24 & 0 \\ 0 & 2/24 & -6/24 & 4/24 \\ 0 & 0 & 3/24 & -3/24 \end{bmatrix} \tag{19}$$

  $\pi_3 = 64/125$ item (7.30a) Mean is 2 item (7.30b) 0.346
- (7.34) 25,600$ per day
- (8.2a) 0.76
- (8.2b) x
- (8.2c) $\sqrt{s/(t+s)}$
- (8.2d) 3
- (8.2e) 0.983
- (8.16) $a = \pm 1/\sqrt{2}$