MVE550 2019 Lecture 12

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Overview

- Properties for the long run; review.
- Time reversible chains
- Queueing theory
- Poisson subordination
- Back to some more examples, if time.

Limiting and stationary distributions

- A probability vector v represents a *limiting distribution* if, for all states i and j,
$$\lim_{t \to \infty} P_{ij}(t) = v_j.$$
- A probability vector v represents a *stationary distribution*, if, for all $t \geq 0$,
$$v = vP(t)$$
- A limiting distribution is a stationary distribution but not necessarily vice versa.
- A continuous-time Markov chain is *irreducible* if for all i and j there exists a $t > 0$ such that $P_{ij}(t) > 0$.
- However, periodic continuous-time Markov chains do not exist: If $P_{ij}(t) > 0$ for some $t > 0$ then $P_{ij}(t) > 0$ for all $t > 0$.

The fundamental limit theorem

- For a finite-state continuous-time Markov chain with finite holding time parameters, there are two possibilities:
  - The process is irreducible, and $P_{ij}(t) > 0$ for all $t > 0$ and all i, j.
  - The process contains one or more absorbing communication classes.
- *Fundamental Limit Theorem*: Let $\{X_t\}_{t\geq 0}$ be a finite, irreducible, continuous-time Markov chain with transition funciton $P(t)$. Then there exists a unique stationary distribution vector v which is also the limiting distribution.
- A probability vector v is a stationary distribution of a continuous-time Markov chain with infinitesimal generator Q if and only if $vQ = 0$.

Absorbing states

- Assume $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain with k states. Assume the last, called a, is absorbing and the rest are not. (They are then transient).
- The entire row for a must consist of zeros. We may write
$$Q = \begin{bmatrix} V & * \\ \mathbf{0} & 0 \end{bmatrix}.$$
- Let F be the $(k-1)\times(k-1)$ matrix so that F_{ij} is the expected time spent in state j when the state starts in i. We can shown that $(VF) = -I$, so that $F = -V^{-1}$.
- Note that, if the chain starts in state i, the expected time until absorbtion is the sum of the i'th row of F.

Stationary distribution of the embedded chain

- Recall the *embedded chain* of a continuous-time Markov chain, with transition matrix $\tilde{P}$.
- Stationary distributions for the embedded chain and for the continuous-time chain are generally not the same!
- However, there is a simple relationship: A probability vector v is a stationary distribution for a continuous-time Markov chain if and only if ψ is a stationary distribution for the embedded chain, where $\psi_j = Cv_j q_j$ for the appropriate normalizing constant C.

Global Balance

- For a continuous-time Markov chain, the long term rate of movement *into* a state must correspond to the long term rate of movement *out of* the chain. This is called *global balance*.
- This corresponds to the equation, for each state j,

$$\sum_{i \neq j} \pi_i q_{ij} = \pi_j q_j$$

- Note that this corresponds exactly to the rows of the matrix equation $\pi Q = 0$, which we know holds for stationary distributions π.
- Generalization: If A is a set of states, then the long term rates of movement *into* and *out of* A are the same:

$$\sum_{i \in A} \sum_{j \notin A} \pi_i q_{ij} = \sum_{i \in A} \sum_{j \notin A} \pi_j q_{ji}$$

Time reversibility and local balance

- The continuous-time Markov chain with unique stationary distribution π is said to be *time reversible* if for all i, j,

$$\pi_i q_{ij} = \pi_j q_{ji}$$

- This is called the *local balance* condition.
- Note: The rate of observed changes from i to j is the same as the rate of observed changes from j to i. Thus we have *time reversibility*.
- Note that (similar to discrete chains): If a probability vector v satisfies local balance condition, then v is the unique stationary distribution. (Easy to show).

Markov processes with transition graphs that are trees

- A *tree* is a graph that does not contain cycles.
- Assume the transition graph of an irreducible continuous-time Markov chain is a tree.
- From the generalized global balance property, it then follows that the process is time reversible, i.e., $\pi_i q_{ij} = \pi_j q_{ji}$ for all i and j.
- Note that the process can be time reversible even if the transition graph is not a tree.

Birth-and-death processes

- A birth-and-death process is a continuous-time Markov chain where the state space is the set of nonnegative integers and transitions only occur to neighbouring integers.
- The process is necessarily time-reversible, as the transition graph is a tree (in fact, a line).
- We denote the rate of *births* from i to $i+1$ with λ_i, and the rate of *deaths* from i to $i-1$ with μ_i.
- The generator matrix is

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\mu_3+\lambda_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Provided $\sum_{k=0}^{\infty}\prod_{k=1}^{\infty}\frac{\lambda_{i-1}}{\mu_i} \leq \infty$, the unique stationary distribution is given by

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}, \text{ and } \pi_0 = \left(\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \right)^{-1}.$$

Queues

- This is a large set of continuous-time stochastic processes with the set of non-negative integers as the state space. Not necessarily Markov.
- A common notation is on the form A/B/c, where A describes the arrival process, B the service time process, and c describes the number of "servers".
- Little's formula:

$$L = \lambda W$$

  - L: Long term average number of "customers" in system.
  - λ: Long term rate of arrival of customers.
  - W: Long term average time for customer in system.

M/M/1

- M means "Markov" (or memoryless): Arrival times and service times have exponential distributions, and there is one server.
- If $\{X_t\}_{t \geq 0}$ denotes the number of customers in the system at time t, then this is a birth-and-death process with constant birth rate λ and constant death rate μ. (Why?)
- Using the formula for the limiting distribution for birth-and-death processes, we can show that for M/M/1 queues, it becomes a geometric distribution with parameter $1 - \lambda/\mu$.

M/M/c

- The arrival times and service times have exponential distributions, but there are now c servers.
- If $\{X_t\}_{t \geq 0}$ denotes the number of customers in the system at time t, then this is a birth-and-death process with
  - The birth rate is constant, λ.
  - The death rate is

$$\mu_i = \begin{cases} \mathrm{i}\mu & \text{for } i = 1, \ldots, c \\ \mathrm{c}\mu & \text{for } i \geq c \end{cases}$$

    for some μ.
- Using the general formula for the limiting distribution for birth-and-death processes, we get that

$$\pi_k = \begin{cases} \frac{\pi_0}{k!} \left(\frac{\lambda}{\mu}\right)^k & \text{for } k = 1, \ldots, c \\ \frac{\pi_0}{c^{k-c} c!} \left(\frac{\lambda}{\mu}\right)^k & \text{for } k \geq c \end{cases}$$

Poisson subortination

- Let $Y_0, Y_1, \ldots,$ be a discrete time discrete state space Markov chain with transition matrix R. Let $\{N_t\}_{t \geq 0}$ be a Poisson process with parameter λ. Then $X_t = Y_{N_t}$ is a continuous time Markov chain with

$$P(t) = \sum_{i=0}^{\infty} R^k \, \mathrm{Poisson}(k; t\lambda). \tag{1}$$

- Conversely, assume X_t is a continuous time Markov chain with generator matrix Q. Define

$$R = \frac{1}{\lambda} Q + I$$

  where λ is chosen so that R has only positive elements. Then Equation 1 holds, so X_t is described as a Poisson subordination process as above.
- The discrete time process and the continuous time process have the same stationary distributions.
- Using Equation 1 for $P(t)$ and truncating the number of terms provides a way to estimate $P(t)$, if $\exp(tQ)$ is difficult to estimate.