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Suggested solutions for
MVE550 Stochastic Processes and Bayesian Inference
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1. (a) A regular transition matrix P is a transition matrix such that there is an $n > 0$ such that P^n is a positive matrix: A positive matrix is one where all the elements are positive.
   (b) A communication class is a subset S of states such that, for all $i, j \in S$, there are $n > 0$ and $m > 0$ such that $P^m_{ij} > 0$ and $P^n_{ij} > 0$, while for any pair $i \in S$ and $j \notin S$, this is not the case. A closed communication class is a communication class with a zero probability of ever leaving the class.
   (c) A state j is transient if the probability that a chain starting at j will ever return to j is less than 1.
   (d) A state j is positive recurrent if the expected number of steps for a chain to return to j if it starts at j is finite.
   (e) A finite state space Markov chain is ergodic if it is irreducible and aperiodic.
   (f) Time reversibility means that, for all states i and j, $\pi_i P_{ij} = \pi_j P_{ji}$.

2. (a) Using Bayes theorem we get

$$\begin{aligned}
\pi(\lambda \mid x) &\propto_\lambda \pi(x \mid \lambda)\pi(\lambda) \\
&\propto_\lambda \text{Exponential}(x; \lambda)\,\text{Gamma}(\lambda; \alpha, \beta) \\
&\propto_\lambda \lambda \cdot \exp(-\lambda x) \cdot \lambda^{\alpha-1} \cdot \exp(-\beta\lambda) \\
&\propto_\lambda \lambda^{\alpha} \cdot \exp(-(\beta + x)\lambda) \\
&\propto_\lambda \text{Gamma}(\lambda; \alpha + 1, \beta + x)
\end{aligned}$$

   In other words, the posterior distribution is a Gamma distribution with parameters $\alpha + 1$ and $\beta + x$.
   (b) The prior corresponds to a Gamma(0, 0) distribution. The posterior is obtained by updating the Gamma distribution as in (a) with the data given, resulting in the posterior

$$\text{Gamma}(3, 1.2 + 1.7 + 0.9) = \text{Gamma}(3, 3.8).$$

3. (a) We get the transition matrix

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.6 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) We get

$$Q = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}$$

and so

$$\begin{aligned} F &= (I-Q)^{-1} = \begin{bmatrix} 0.9 & -0.3 \\ -0.1 & 0.8 \end{bmatrix}^{-1} = \frac{1}{0.9\cdot 0.8 - 0.3\cdot 0.1}\begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} \\ &= \frac{1}{0.69}\begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} = \begin{bmatrix} 1.1594 & 0.4348 \\ 0.1449 & 1.3043 \end{bmatrix}. \end{aligned}$$

(c) We have

$$R = \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.4 \end{bmatrix}$$

and thus

$$FR = \frac{1}{0.69}\begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.9 \end{bmatrix}\begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.4 \end{bmatrix} = \frac{1}{0.69}\begin{bmatrix} 0.57 & 0.12 \\ 0.33 & 0.36 \end{bmatrix}.$$

Thus the probability for a process that starts in state 1 to be absorbed in state 4 is $\frac{0.12}{0.69} = 0.1739$.

4. Assume $X_0, X_1, \dots, X_n, \dots$ is an ergodic Markov chain with stationary distribution π. Let r be a bounded real-valued function. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} r(X_i) = \mathrm{E}(r(X))$$

where X is a random variable with distribution π.

5. The holding time parameters are $q = (2, 3, 4)$. The embedded chain transition matrix is

$$\tilde{P} = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.9 & 0 & 0.1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus the generator matrix becomes

$$Q = \begin{bmatrix} -2 & 1.2 & 0.8 \\ 2.7 & -3 & 0.3 \\ 4 & 0 & -4 \end{bmatrix}.$$

The linear system $\pi Q = 0$ gives

$$\begin{aligned} -2\pi_1 + 2.7\pi_2 + 4\pi_3 &= 0 \\ 1.2\pi_1 - 3\pi_2 &= 0 \\ 0.8\pi_1 + 0.3\pi_2 - 4\pi_3 &= 0 \end{aligned}$$

with solution $\pi = \frac{1}{163}(100, 40, 23)$. Thus, the long-term proportion of time that the component spends in state A is $100/163 = 0.6135$.

6. A hidded Markov model consists of a Markov chain $X_1, X_2, \dots, X_n$ of "hidden" random variables, and another sequence $Y_1, \dots, Y_n$ of variables such that the distribution of Y_i only depends on X_i, and possibly on Y_{i-1}. Thes latter variables are the "observed" variables. If the values of the variables Y_i are indeed observed, the posterior distribution for one of the hidden variables, say X_i can be found as follows: In a "Forward" part of the algorithm, for $j = 1, \dots, i$, the posterior for X_j given $Y_1, \dots, Y_j$ is found in a recursive algorithm. In a "Backward" part of the algorithm, for $j = n, \dots, i$, the likelihoods for X_j given the data $Y_j, \dots, Y_n$ are found in a recursive algorithm. Then the two are put together to find the marginal posterior for X_i.

7. (a) The events that three requests of type A arrive during the time unit and that four requests of type B arrive during the time unit are independent, and the probability of both can be computed using the Poisson probability mass function. Thus the answer is
$$e^{-\lambda_A}\frac{\lambda_A^3}{3!}e^{-\lambda_B}\frac{\lambda_B^4}{4!} = e^{-3}\frac{3^3}{3!}e^{-2}\frac{2^4}{4!} = 3e^{-5} = 0.02021384.$$
   (b) The probability of the first event being a request of type A or B is $\frac{\lambda_A}{\lambda_A+\lambda_B}$, respectively $\frac{\lambda_B}{\lambda_A+\lambda_B}$. As the successive events are independent, the probability asked for is
$$\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^3\left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^4 = \frac{\lambda_A^3\lambda_B^4}{(\lambda_A+\lambda_B)^7}.$$

8. We have
$$\mathrm{E}(X_t) = a + b(\mathrm{E}(B_t) + \mathrm{E}(W_{3+t})) = a + b(0 + W_3).$$
Setting this to zero gives $a + bW_3 = 0$. Further,
$$\mathrm{Var}(X_t) = b^2\left(\mathrm{Var}(B_t) + \mathrm{Var}(W_{3+t})\right) = b^2(t+t)$$
and setting this to t gives $2b^2 = 1$. Thus we must have $b = \frac{1}{\sqrt{2}}$ and $a = -\frac{1}{\sqrt{2}}W_3$. On the other hand, with these values,
$$a + b(B_t + W_{3+t}) = \frac{1}{\sqrt{2}}(B_t + (W_{3+t} - W_3))$$
fulfills all criteria for a Brownian motion.