MVE550 2019 Lecture 13

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Overview

- Definition of standard Brownian motion.
- Simulation of Brownian motion.
- Computing with Brownian motion.
- Connection to random walks.
- Gaussian processes.
- First hitting time.
- Maximum of Brownian motion.
- Zeros of Brownian motion.

Continuous-time continuous state space processes

- Having looked at
  - Discrete-time discrete state space processes.
  - Discrete-time continuous state space processes (e.g., some MCMC examples).
  - Continuous-time discrete state space processes (e.g., Poisson processes and more generally continuous-time Markov chains).

  we now look at continuous-time continuous state space processes.
- We will look at two examples:
  - Brownian motion.
  - More generally, Gaussian processes.

(Standard) Brownian motion

Standard Brownian motion is a continuous-time stochastic process $\{B_t\}_{t \geq 0}$ with the following defining properties:

1. $B_0 = 0$.
2. For $t > 0$, $B_t \sim \text{Normal}(0, t)$ (so the *variance* is t).
3. For $s, t > 0$, $B_{t+s} - B_s \sim \text{Normal}(0, t)$.
4. For $0 \leq q < r \leq s < t$, $B_t - B_s$ is independent from $B_r - B_q$.
5. The function $t \mapsto B_t$ is continuous with probability 1.

Simulation of Brownian motion

- Given time points $t_1 < t_2 < \cdots < t_n$, we can write

$$B_{t_i} = B_{t_{i-1}} + (B_{t_i} - B_{t_{i-1}}) = B_{t_{i-1}} + Z_i$$

  where $Z_i \sim \text{Normal}(0, t_i - t_{i-1})$.
- Writing $t_0 = 0$, we get

$$B_{t_n} = \sum_{i=1}^{n} Z_i.$$

- A good way to simulate the path $t \mapsto B_t$ on $t \in [0, a]$ is to set $t_i = ai/n$, simulate independently

$$Z_i \sim \text{Normal}(0, a/n)$$

  and compute

$$B_{t_i} = \sum_{j=1}^{i} Z_j.$$

- Note that we can also write $Z_i = \sqrt{a/n}Y_i$, where $Y_i \sim \text{Normal}(0, 1)$.

Nowhere differentiable paths

- We have seen in our simulations that paths of Brownian motion are "jagged".
- We have also seen that this quality is unchanged if we change the scale, i.e., look at smaller intervals.
- Using these observations as starting points, one may show that the path (i.e., the function $t \mapsto B_t$) of a Brownian motion is nowehere differentiable.

Computing with Brownian motion

- Show that $B_1 + B_3 + 2B_7 \sim \text{Normal}(0, 50)$.
- Show that $P(B_2 > 0 \mid B_1 = 1) = 0.8413$.
- Show that $\text{Cov}(B_s, B_t) = \min\{s, t\}$.

Connection to random walks

- We saw above how B_t can be expressed as a large sum of independent normal random variables.
- If we replace the random variables with others which also have zero expectation and the same variance, we get (approximately) the same result (remember the Central Limit Theorem)!
- If we multiply the number of variables with k and scale the output with $1/\sqrt{k}$, the result is (more or less) unchanged.
- One can use this effect to study the limiting behaviour of a random walk with a Brownian motion: When $k \to \infty$ the two processes are the same. Results for Brownian motion may be easier to compute.
- This is called the invariance principle.

Remember: The multivariate normal distribution

- Definition (one of many): A set of random variables $X_1, \ldots, X_k$ has a *multivariate normal distribution* if, for all real $a_1, \ldots, a_k$, $a_1 X_1 + \cdots + a_k X_k$ is normally distributed.
- It is completely determined by the expectation vector $\mu = (\mathrm{E}(X_1), \ldots, \mathrm{E}(X_k))$ and the $(k \times k)$ covariance matrix Σ, where $\Sigma_{ij} = \mathrm{Cov}(X_i, X_j)$.
- The joint density function on the vector $x = (x_1, \ldots, x_k)$ is

$$\pi(x) = \frac{1}{|2\pi\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right).$$

  where $|2\pi\Sigma|$ is the determinant of the matrix $2\pi\Sigma$.
- All marginal distributions and all conditional distributions are also multivariate normal.

Gaussian processes

- A *Gaussian process* is a continuous-time stochastic process $\{X_t\}_{t\geq 0}$ with the property that for all $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$, $X_1, \ldots, X_n$ has a multivariate normal distribution.
- Thus, a Gaussian process is completely determined by its mean function $\mathrm{E}(X_t)$ and its covariance function $\mathrm{Cov}(X_s, X_t)$.
- Gaussian processes are extremely versatile as models. One may generalize for example so that the index set the t's belong to is $\mathbb{R}^n$.

Brownian motion and Gaussian processes

- Brownian motion is a Gaussian process, as we can show that any $a_1 B_{t_1} + \cdots + a_k B_{t_k}$ is normally distributed.
- A Gaussian process $\{X_t\}_{t \geq 0}$ is Brownian motion if it fulfills
  1. $X_0 = 0$.
  2. $\mathrm{E}(X_t) = 0$ for all t.
  3. $\mathrm{Cov}(X_s, X_t) = \min\{s, t\}$ for all s, t.
  4. The function $t \mapsto X_t$ is a continuous with probability 1.
- It is mostly easy to show that the following tranformations of a Brownian motion $\{B_t\}_{t \geq 0}$ are Brownian motions (show that they are Gaussian processes fulfilling the criteria above):
  - $\{-B_t\}_{t \geq 0}$.
  - $(B_{t+s} - B_s)_{t \geq 0}$ for any $s \geq 0$.
  - $\left\{ \frac{1}{\sqrt{a}} B_{at} \right\}_{t \geq 0}$ for any $a > 0$.
  - The process $\{X_t\}_{t \geq 0}$ where $X_0 = 0$ and $X_t = t B_{1/t}$ for $t > 0$.
- The process $X_t = x + B_t$ where B_t is Brownian motion and x is some real number is called "Brownian motion started at x".

First Hitting Time

- The *first hitting time* T_a is defined as $T_a = \min\{t : B_t = a\}$.
- It can be shown that $B_{t+T_a} - a$ is Brownian motion (i.e., that T_a is a "stopping time").
- It follows that (for $a > 0$)

$$\Pr\left(B_t > a \mid T_a < t\right) = \Pr\left(B_{t-T_a} > 0\right) = \frac{1}{2}.$$

- We get that (for $a > 0$)

$$\Pr\left(T_a < t\right) = 2\Pr\left(B_t > a\right)$$

- We get that the density on $t \in (0, \infty)$ is given (for $a \neq 0$) by

$$\pi(t) = \frac{|a|}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right).$$

- This means that $t \sim \text{Inverse-Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$, i.e., $\frac{1}{t} \sim \text{Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$.

Maximum of Brownian motion

- Define $M_t = \max_{0 \le s \le t} B_s$.
- We may compute (using previous overhead) for $a > 0$

$$\Pr(M_t > a) = \Pr(T_a < t) = 2\Pr(B_t > a) = \Pr(|B_t| > a)$$

- Thus M_t has the same distribution as $|B_t|$, the absolute value of B_t.
- Example: Find t such that $\Pr(M_t \le 4) = 0.9$.

Zeros of Brownian motion

- Theorem: The probability that Brownian motion has at least one zero in (r, t), with $0 \leq r < t$, is

$$z_{r,t} = \frac{2}{\pi} \arccos\left(\sqrt{\frac{r}{t}}\right).$$

- Proof uses the distribution of M_t.
- The probability can be written `1 - pbeta(r/t, 0.5, 0.5)`.
- Let L_t be the last zero in $(0, t)$. Then

$$P(L_t \leq x) = 1 - z_{x,t} = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{x}{t}}\right).$$

  which can be computed as `pbeta(x/t, 0.5, 0.5)`.
- In other words, the last zero is distributed so that $x/t \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$.