MVE550 2019 Lecture 14

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Overview

- Stuff we did not cover last time:
  - Finish: First Hitting Time
  - Maximum of Brownian motion
  - Zeros of Brownian motion
- Variations of Brownian motion.
- Martingales.
- Geometric Brownian motion. Stock options.
- A short review of the course (with a view to the exam)
- A very short discussion about / evaluation of the course.

First Hitting Time

- The *first hitting time* T_a is defined as $T_a = \min\{t : B_t = a\}$.
- It can be shown that $B_{t+T_a} - a$ is Brownian motion (i.e., that T_a is a "stopping time").
- It follows that (for $a > 0$)

$$\Pr(B_t > a \mid T_a < t) = \Pr(B_{t-T_a} > 0) = \frac{1}{2}.$$

- We get that (for $a > 0$)

$$\Pr(T_a < t) = 2\Pr(B_t > a)$$

- We get that the density on $t \in (0, \infty)$ is given (for $a \neq 0$) by

$$\pi(t) = \frac{|a|}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right).$$

- This means that $t \sim \text{Inverse-Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$, i.e., $\frac{1}{t} \sim \text{Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$.

Maximum of Brownian motion

- Define $M_t = \max_{0 \leq s \leq t} B_s$.
- We may compute (using previous overhead) for $a > 0$

$$\Pr(M_t > a) = \Pr(T_a < t) = 2\Pr(B_t > a) = \Pr(|B_t| > a)$$

- Thus M_t has the same distribution as $|B_t|$, the absolute value of B_t.
- Example: Find t such that $\Pr(M_t \leq 4) = 0.9$.

Zeros of Brownian motion

- Theorem: The probability that Brownian motion has at least one zero in (r, t), with $0 \leq r < t$, is

$$z_{r,t} = \frac{2}{\pi} \arccos\left(\sqrt{\frac{r}{t}}\right).$$

- Proof uses the distribution of M_t.
- The probability can be written `1 - pbeta(r/t, 0.5, 0.5)`.
- Let L_t be the last zero in $(0, t)$. Then

$$P(L_t \leq x) = 1 - z_{x,t} = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{x}{t}}\right).$$

  which can be computed as `pbeta(x/t, 0.5, 0.5)`.
- In other words, the last zero is distributed so that $x/t \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$.

Brownian motion with a drift, Brownian bridge

- For real μ and $\sigma > 0$ define the Gaussian process X_t as

$$X_t = \mu t + \sigma B_t$$

  This is *Brownian motion with a drift*, and is often a more useful model than standard Brownian motion.
- Define a Gaussian process X_t by conditioning Brownian motion B_t on $B_1 = 0$. Then X_t is a *Brownian bridge*.
- Results for multivariate normal distributions can be used to derive properties, such as $\text{Cov}(X_s, X_t) = \min(s, t) - st$.
- In fact, a Brownian bridge can be expressed as $B_t - tB_1$, which makes it easy to simulate.

Geometric Brownian motion

- The stochastic process

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  where $G_0 > 0$ is called *geometric Brownian motion* with drift parameter μ and variance σ^2.
- $\log(G_t)$ is a Gaussian process with expectation $\log(G_0) + \mu t$ and variance $t\sigma^2$.
- One can show that $\mathrm{E}(G_t) = G_0 e^{t(\mu+\sigma^2/2)}$ and $\mathrm{Var}(G_t) = G_0^2 e^{2t(\mu+\sigma^2/2)}(e^{t\sigma^2} - 1)$.
- Natural model for things that develop by multiplication of random independent factors, rather than addition of random independent increments. Example: Stock prices.
- Option: The right (but not obligation) to buy specific stock at a fixed future time at a fixed price.
- To compute the value of an option: Using a geometric Brownian motion model, calculate the expected value of the stock at the future date conditional on the value being above the agreed price.

Martingales

- A stochastic process $(Y_t)_{t\geq 0}$ is a *martingale* if for $t \geq 0$
  - $\mathrm{E}(Y_t \mid Y_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) \leq \infty$.
- $(Y_t)_{t\geq 0}$ is a *martingale with respec to* $(X_t)_{t\geq 0}$ if for all $t \geq 0$
  - $\mathrm{E}(Y_t \mid X_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) \leq \infty$.
- Brownian motion is a martingale.
- Example: If $G_t = G_0 e^{\mu t + \sigma B_t}$ is geometric Brownian motion, then

$$e^{-(\mu+\sigma^2/2)t} G_t$$

is a martingale with respect to standard Brownian motion.

The Black-Scholes formula for option pricing

- An important application of Geometric Brownian motion models.
- Based on assuming that the rate r of depreciation of money ("risk free investment") r is equal to $\mu + \sigma^2/2$, so that $e^{-rt}G_t$ is a martingale for a stock.
- Use the previously mentioned way of computing the value of a stock option, adjusted to a present value using e^{-rt}.