Exponential generating functions.

Let $\{a_n\}_0^\infty$ be a series. The exponential generating function of $\{a_n\}_0^\infty$ is defined by

$$E(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} .$$

Rmk: If $a_n = 1$ for all $n \geq 0$, then $E(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ with is equal to e^t, therefore the name "exponential generating function".

Moment generating functions

Let X be a random variable. Recall that the moments of X are:

$$E[X^k] = \sum x_i^k \, p(X = x_i)$$

(the expected value of X^k).

The moment generating function of X is $E[e^{tX}]$, the expected value of e^{tX}.

Consider now the exponential generating function $E(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ and let $a_n = E[X^n]$ for some random variable X. Then $E(t) = \sum_{n=0}^{\infty} E(X^n) \frac{t^n}{n!}$.

Expand $E(t)$:

$$E(t) = 1 + E[X]\frac{t}{1!} + E[X^2]\frac{t^2}{2!} + E[X^3]\frac{t^3}{3!} + \cdots$$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right]$$

$$= E\left[e^{tX}\right] = m_X(t).$$

Recall that the Mc. Laurin series of $m_X(t)$ is

$$m_X(t) = \sum_{n=0}^{\infty} \frac{m_X^{(n)}(0)}{n!} t^n.$$

Therefore, $m_X'(0) = E[X]$

$m_X''(0) = E[X^2]$

$m_X'''(0) = E[X^3]$

$\vdots$

$m_X^{(n)}(0) = E[X^n]$.

Ex: Binomial distribution $Bin(n,p)$

Density fn: $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$m_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$

$$= (e^t p + 1 - p)^n.$$

$$m_X'(t) = n(e^t p + 1 - p)^{n-1} e^t p$$

$$m_X''(t) = n(n-1)(e^t p + 1 - p)^{n-2} (e^t p)^2 + n(e^t p + 1 - p)^{n-1} e^t p.$$

Hence, $E[X] = m_X'(0) = n(p+1-p)^{n-1}p = np$

$$E[X^2] = m_X''(0) = n(n-1)\underbrace{(p+1-p)^{n-2}}_{1}p^2 + n\underbrace{(p+1-p)^{n-1}}_{1}p$$

$$= n(n-1)p^2 + np$$

$$\Rightarrow V[X] = m_X''(0) - (m_X'(0))^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p)$$

Thm 1: Let X and Y be two random variables with moment generating functions $m_X(t)$ and $m_Y(t)$ respectively. If $m_X(t) = m_Y(t)$ for all t in some open interval about 0, then X and Y have the same distribution.

Thm 2: Let X be a random variable with moment generating function $m_X(t)$. Let $Y = \alpha + \beta X$. The moment generating function for Y is

$$m_Y(t) = e^{\alpha t} m_X(\beta t).$$

We use Thm 1 and Thm 2 to prove that if $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim N(0,1)$.

Note first that if $X \sim N(\mu, \sigma^2)$, then $m_X(t) = e^{\mu t + \sigma^2 t^2/2}$.

Let $Z \sim N(0,1)$, then $m_Z(t) = e^{t^2/2}$.

$$m_{\frac{X-\mu}{\sigma}}(t) = m_{-\frac{\mu}{\sigma} + \frac{X}{\sigma}}(t) = e^{-\frac{\mu}{\sigma}t} m_X\left(\frac{1}{\sigma}t\right) \qquad \text{(Thm 2)}$$

$$= e^{-\frac{\mu}{\sigma}t} e^{\mu \cdot \frac{1}{\sigma}t + \sigma^2 \frac{1}{\sigma^2} t^2/2}$$

$$= e^{t^2/2} = m_Z(t)$$

Hence, since $m_{\frac{X-\mu}{\sigma}}(t) = m_Z(t)$ for all t, $\frac{X-\mu}{\sigma}$ and Z have the same distribution, and $\frac{X-\mu}{\sigma} \sim N(0,1)$.

<u>Thm 3</u> Let X_1 and X_2 be independent random variables with moment generating functions $m_{X_1}(t)$ and $m_{X_2}(t)$ respectively. Let $Y = X_1 + X_2$. The moment generating function of Y is given by

$$m_Y(t) = m_{X_1}(t) \cdot m_{X_2}(t).$$

From Thm 3 we deduce the following:

- If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

<u>Pf:</u>
$$m_{X_1+X_2}(t) = m_{X_1}(t) \cdot m_{X_2}(t)$$
$$= e^{\mu_1 t + \sigma_1^2 t^2/2} \cdot e^{\mu_2 t + \sigma_2^2 t^2/2}$$
$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}.$$

- If $X_1 \sim Poiss(\lambda_1)$ and $X_2 \sim Poiss(\lambda_2)$ are indep. then $X_1 + X_2 \sim Poiss(\lambda_1 + \lambda_2)$.

<u>Pf:</u> Note first that $m_{X_1}(t) = e^{\lambda_1(e^t-1)}$ and $m_{X_2}(t) = e^{\lambda_2(e^t-1)}$. Therefore,

$$m_{X_1+X_2}(t) = m_{X_1}(t) \cdot m_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}.$$