(A) Solution to 23-25-29 – Section 13.2

23 $\begin{cases} a_0 = 8 \\ a_1 = 16 \\ a_n = 2a_{n-1} + 3a_{n-2} \quad n>1 \end{cases}$

$$\sum_{n>1} a_n x^n = 2\sum_{n>1} a_{n-1} x^n + 3\sum_{n>1} a_{n-2} x^n$$

$$= 2x \sum_{n>1} a_{n-1} x^{n-1} + 3x^2 \sum_{n>1} a_{n-2} x^{n-2}$$

$$= 2x \sum_{n>0} a_n x^n + 3x^2 \sum_{n\geq 0} a_n x^n$$

$$\Rightarrow \sum_{n\geq 0} a_n x^n - 8 - 16x = 2x\left(\sum_{n\geq 0} a_n x^n - 8\right) + 3x^2 \sum_{n\geq 0} a_n x^n$$

$$f(x) - 8 - 16x = 2x\, f(x) - 16x + 3x^2 f(x)$$

$$\Rightarrow f(x) - 2x f(x) - 3x^2 f(x) = 8$$

$$\Rightarrow f(x)(1-2x-3x^2) = 8 \quad \Rightarrow f(x) = \frac{8}{1-2x-3x^2}$$

$$1-2x-3x^2 = (1+x)(1-3x)$$

$$\Rightarrow \frac{8}{1-2x-3x^2} = \frac{A}{1+x} + \frac{B}{1-3x} = \frac{A - 3Ax + B + Bx}{(1+x)(1-3x)} = \frac{(A+B) + (-3A+B)x}{(1+x)(1-3x)}$$

$$\Rightarrow \begin{cases} A+B=8 \quad (1) \\ -3A+B=0 \end{cases} \Rightarrow B = 3A \overset{(1)}{\Rightarrow} A+3A=8 \Rightarrow 4A=8 \Rightarrow A=2$$

$$\Rightarrow B = 6$$

$$\Rightarrow f(x) = \frac{2}{1+x} + \frac{6}{1-3x} = 2\sum_{n\geq 0}(-x)^n + 6\sum_{n\geq 0}(3x)^n$$

$$= \sum_{n\geq 0}\left(2\cdot(-1)^n + 6\cdot 3^n\right)x^n$$

$$\Rightarrow \boxed{a_n = 2\cdot(-1)^n + 6\cdot 3^n}$$

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$$\begin{cases} a_0 = 4 \\ a_n = 2a_{n-1} - 3 & n>0 \end{cases}$$

$$\sum_{n>0} a_n x^n = 2\sum_{n>0} a_{n-1} x^n - 3\sum_{n>0} x^n$$

$$\Rightarrow \sum_{n\geq 0} a_n x^n - 4 = 2x \sum_{n>0} a_{n-1} x^{n-1} - 3x \sum_{n>0} x^{n-1}$$

$$= 2x \sum_{n\geq 0} a_n x^n - 3x \sum_{n\geq 0} x^n$$

$$f(x) - 4 = 2x\, f(x) - 3x \cdot \frac{1}{1-x}$$

$$\Rightarrow f(x) - 2x\, f(x) = 4 - \frac{3x}{1-x} = \frac{4-4x-3x}{1-x} = \frac{4-7x}{1-x}$$

$$\Rightarrow f(x)(1-2x) = \frac{4-7x}{1-x} \Rightarrow f(x) = \frac{4-7x}{(1-x)(1-2x)}$$

$$f(x) = \frac{A}{1-x} + \frac{B}{1-2x} = \frac{A - 2Ax + B - Bx}{(1-x)(1-2x)} = \frac{(A+B) + (-2A-B)x}{(1-x)(1-2x)}$$

$$\Rightarrow \begin{cases} A+B = 4 \\ -2A - B = -7 \end{cases} \Rightarrow B = 4 - A$$

$$\Rightarrow -2A - 4 + A = -7$$

$$\Rightarrow -A = -3 \Rightarrow A = 3 \Rightarrow B = 1$$

$$f(x) = \frac{3}{1-x} + \frac{1}{1-2x} = 3\sum_{n\geq 0} x^n + \sum_{n\geq 0} (2x)^n$$

$$= \sum_{n\geq 0} (3 + 2^n) x^n \Rightarrow \boxed{a_n = 3 + 2^n}$$

$\underline{\underline{29}}$

$$\begin{cases} a_0 = 1 \\ a_1 = 3 \\ a_n = 7a_{n-1} - 10a_{n-2} & n > 1 \end{cases}$$

$$\sum_{n>1} a_n x^n = 7\sum_{n>1} a_{n-1} x^n - 10\sum_{n>1} a_{n-2} x^n$$

$$= 7x \sum_{n>1} a_{n-1} x^{n-1} - 10x^2 \sum_{n>1} a_{n-2} x^{n-2}$$

$$\Rightarrow \sum_{n\geq 0} a_n x^n - 1 - 3x = 7x \sum_{n>0} a_n x^n - 10x^2 \sum_{n\geq 0} a_n x^n$$

$$= 7x\left(\sum_{n\geq 0} a_n x^n - 1\right) - 10x^2 \sum_{n\geq 0} a_n x^n$$

$$\Rightarrow f(x) - 1 - 3x = 7x\,(f(x) - 1) - 10x^2 f(x)$$

$$= 7x\,f(x) - 7x - 10x^2 f(x)$$

$$\Rightarrow f(x) - 7x\,f(x) + 10x^2 f(x) = 1 + 3x - 7x = 1 - 4x$$

$$\Rightarrow f(x)\left(1 - 7x + 10x^2\right) = 1 - 4x$$

$$\Rightarrow f(x) = \frac{1-4x}{(1-7x+10x^2)} = \frac{1-4x}{(1-5x)(1-2x)} = \frac{A}{1-5x} + \frac{B}{1-2x}$$

$$= \frac{A + B - (2A + 5B)x}{(1-5x)(1-2x)} \Rightarrow \begin{cases} A + B = 1 \\ 2A + 5B = 4 \end{cases} \Rightarrow \begin{matrix} A = \frac{1}{3} \\ B = \frac{2}{3} \end{matrix}$$

$$\Rightarrow f(x) = \frac{1}{3} \cdot \frac{1}{1-5x} + \frac{2}{3} \cdot \frac{1}{1-2x}$$

$$= \frac{1}{3} \sum_{n\geq 0} (5x)^n + \frac{2}{3} \sum_{n\geq 0} (2x)^n = \sum_{n\geq 0} \left(\frac{1}{3} 5^n + \frac{2}{3} 2^n\right) x^n$$

$$\Rightarrow \boxed{a_n = \frac{1}{3} 5^n + \frac{2}{3} \cdot 2^n}$$