

Lecture 12

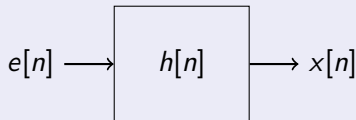
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After today's lecture you should be able to

- Sketch the **PSD** of an **AR**, **MA** and **ARMA** model given poles and zeros of the transfer function.
- Calculate the **ACF** for an **MA** process.
- Derive the **Yule-Walker** equations used to compute the **ACF** of an **AR** process and estimate an **AR** process from data.

A stochastic model for $x[n]$



where $e[n]$ is WSS, white noise,

$$\Rightarrow \begin{cases} E\{e[n]\} = 0 \\ r_e[k] = E\{e[n]e[n+k]\} = \delta[k]\sigma_e^2 = \begin{cases} \sigma_e^2 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

- It follows that

$$P_e(e^{j\omega}) = DTFT\{r_e[k]\} = \sigma_e^2 DTFT\{\delta[k]\} = \sigma_e^2.$$

- The filter $H(z)$ is used to shape the autocorrelation/spectrum

$$\text{of the process } x[n], \quad P_x(e^{j\omega}) = |H(e^{j\omega})|^2 \sigma_e^2$$

- $x[n]$ is an ARMA(p,q)-process if

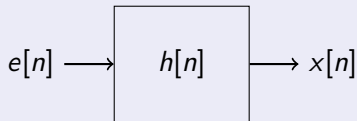
$$H(z) = \frac{B(z)}{A(z)} = \frac{1 + b_1 z^{-1} + \dots + b_q z^{-q}}{1 + a_1 z^{-1} + \dots + a_p z^{-p}}$$

or, equivalently, if

$$x[n] + \dots + a_p x[n-p] = e[n] + \dots + b_q e[n-q] \quad (1)$$

- Important special cases:
 - 1) $x[n]$ is AR(p) if $B(z) = 1$,
 - 2) $x[n]$ is MA(q) if $A(z) = 1$

The PSD of an ARMA



- We get

$$P_x(e^{j\omega}) = P_e(e^{j\omega}) |H(e^{j\omega})|^2 = \sigma_e^2 |H(e^{j\omega})|^2$$

$\leadsto$ the filter $H(z)$ shapes the PSD of $x[n]$.

- **Note:** $e[n]$ is white noise

$$\Rightarrow r_e[n] = \delta[n]\sigma_e^2 \longleftrightarrow P_e(e^{j\omega}) = \sigma_e^2.$$

- **Techniques to compute the ACF:**

- 1) Find the PSD as above and compute

$$r_x[n] = IDTFT \{ P_x(e^{j\omega}) \}$$

↪ general solution, but often rather complicated.

- 2) Derive $h[n] = IDTFT (H(e^{j\omega}))$ and compute

$$r_x[n] = \sigma_e^2 h[n] \star h[-n] = \sum_{k=-\infty}^{\infty} h[k] h[k-n]$$

↪ another possible, but complicated, alternative.

- 3) Derive linear equations from the difference equation (1)
 - ① ↪ a simple solution!
 - ② Only works for AR and MA processes.
 - ③ Enables us to estimate AR-parameters from data.

- Consider an AR(p)-process

$$x[n] + a_1x[n-1] + \cdots + a_px[n-p] = e[n]$$

- Multiply by $x[n-k]$, $k \geq 0$, and take expectations:

Yule-Walker (YW) equations

$$r_x[k] + a_1r_x[k-1] + \cdots + a_pr_x[k-p] = \sigma_e^2 \delta[k]$$

- Note:** 1) YW are linear in the autocorrelation function, $r_x[n]$
2) by changing k , we can collect any number of equations.
 $\Rightarrow$ easy to find $r_x[n]$ using YW