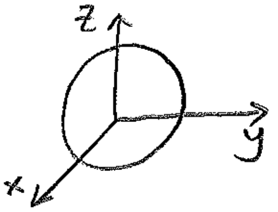
Lösningsförslag till tentan för MVE470/MVE351
den 12 juni 2019

Uppg. 1a



cirkel i planet $x+y+z=1$
med centrum i $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
(och radie $\sqrt{\frac{11}{3}}$)

Uppg 1b

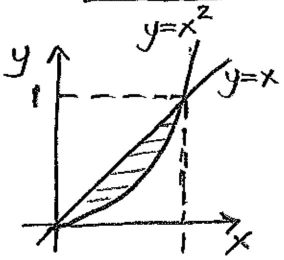
$\nabla f = (y,x)$, $\nabla f(2,0) = (0,2)$

$D_{\mathbf{v}} f(2,0) = (v_1,v_2)\cdot(0,2) = 2v_2 = -1 \Rightarrow v_2 = \frac{-1}{2}$

$v_1^2 + v_2^2 = 1 \Rightarrow v_1 = \pm\sqrt{1-v_2^2} = \pm\sqrt{1-\frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$

Svar: I riktningarna $\mathbf{v} = (\frac{\pm\sqrt{3}}{2}, \frac{-1}{2})$

Uppg 1c



$$\iint_R \frac{x}{y}e^y\,dA = \int_0^1\left(\int_y^{\sqrt{y}} \frac{x}{y}e^y\,dx\right)dy = \int_0^1 \frac{1}{y}e^y\left[\frac{x^2}{2}\right]_y^{\sqrt{y}}dy =$$

$$= \frac{1}{2}\int_0^1 e^y(1-y)\,dy = \frac{1}{2}\left(\left[e^y(1-y)\right]_0^1 + \int_0^1 e^y\,dy\right) = \frac{1}{2}(e-2)$$

Svar: $\frac{1}{2}(e-2)$

Uppg. 1d C kan parametriseras av $\mathbf{r} = t(3,1,-2)$, $0 \xrightarrow{t} 1$

$$\int_C x^2\,ds = \int_0^1 (3t)^2\sqrt{3^2+1^2+(-2)^2}\,dt = 9\sqrt{14}\left[\frac{t^3}{3}\right]_0^1 = 3\sqrt{14}$$

Svar: $3\sqrt{14}$

Uppg 2 a) $f = 3xy - x^2y - xy^2$, $f(1,1) = 1$

$f_1 = 3y - 2xy - y^2$, $f_1(1,1) = 0$

$f_2 = 3x - x^2 - 2xy$, $f_2(1,1) = 0$

$f_{11} = -2y$, $f_{11}(1,1) = -2$

$f_{12} = 3 - 2x - 2y$, $f_{12}(1,1) = -1$

$f_{22} = -2x$, $f_{22}(1,1) = -2$

$$P_2(x,y) = 1 + \frac{1}{2}\left(-2(x-1)^2 - 2(x-1)(y-1) - 2(y-1)^2\right) =$$
$$= \underline{\underline{1 - (x-1)^2 - (x-1)(y-1) - (y-1)^2}}$$

Vi noterar spec. att;

$$P_2(x,y) = 1 - \left(\left((x-1) + \frac{1}{2}(y-1)\right)^2 + \frac{3}{4}(y-1)^2\right) < 1 = f(1,1)$$

för alla $(x,y) \neq (1,1)$.
så för alla (x,y) i någon omgivning B av $(1,1)$
måste även $f(x,y) \leq f(1,1)$
dvs. $f(x,y)$ har ett $\underline{\underline{\text{lokalt max i } (1,1)}}$

Detta kan också inses genom att studera funktionens Hessian-matris i $(1,1)$;

$$\mathcal{H}(1,1) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}, \quad \left.\begin{array}{l} \det \mathcal{H}(1,1) = 3 > 0 \\ f_{11}(1,1) < 0 \end{array}\right\} \Rightarrow \text{lok. max.}$$

b) $$\begin{cases} f_1 = 3y - 2xy - y^2 = y(3 - 2x - y) = 0 \\ f_2 = 3x - x^2 - 2xy = x(3 - x - 2y) = 0 \end{cases}$$

Vi avläser att f har de stationära punkterna $(0,0), (1,1), (0,3), (3,0)$, varav alla utom $(0,3)$ ligger i D. Vidare är

$\underline{f(0,0) = 0}$, $\underline{f(1,1) = 1}$, $\underline{f(3,0) = 0}$

f saknar singulära punkter.

Vi undersöker även de fyra randbitarna;

$x=0, 0<y<2$: $\underline{f(0,y)=0}$

$y=0, 0<x<4$: $\underline{f(x,0)=0}$

$x=4, 0<y<2$: $f(4,y)=12y-16y-4y^2=-4y-4y^2=g(y)$

$g'(y)=-4-8y=0 \Leftrightarrow y=\frac{-1}{2}$ ← ligger inte i intervallet $0\leq y\leq 2$

$y=2, 0<x<4$: $f(x,2)=6x-2x^2-4x=2x-2x^2=h(x)$

$h'(x)=2-4x=0 \Leftrightarrow x=\frac{1}{2}$ ← ligger i intervallet $0\leq x\leq 4$

$\underline{f(\tfrac{1}{2},2)=\tfrac{1}{2}}$

Vi undersöker även "hörnen";

$\underline{f(0,0)=0}$, $\underline{f(0,2)=0}$, $\underline{f(4,0)=0}$, $\underline{f(4,2)=-24}$

Svar: Det största och minsta värdet är $\underline{\underline{1}}$ resp. $\underline{\underline{-24}}$

Uppg. 3

$$r^2=\sqrt{2-r^2} \Rightarrow r^4+r^2-2=0 \Rightarrow r=\frac{-1}{2}(\pm)\sqrt{\frac{1}{4}+2}=1$$

$$\iiint_R z\,dV=\int_0^{2\pi}\left(\int_0^1\left(\int_{r^2}^{\sqrt{2-r^2}} z\,r\,dz\right)dr\right)d\theta=$$

$$=2\pi\int_0^1 r\left[\frac{z^2}{2}\right]_{r^2}^{\sqrt{2-r^2}}dr=\pi\int_0^1 r(2-r^2-r^4)\,dr=$$

$$=\pi\left[r^2-\frac{r^4}{4}-\frac{r^6}{6}\right]_0^1=\pi\left(1-\frac{1}{4}-\frac{1}{6}\right)=\underline{\underline{\frac{7\pi}{12}}}$$

Uppg 4b $\mathbb{F}$ är konservativt med potential $\phi=xyz$ så;

$$\int_C \mathbb{F}\cdot d\mathbb{r}=\phi(1,0,0)-\phi(-1,0,0)=0-0=\underline{\underline{0}}$$

alt. kan vi parametrisera C genom

$$\mathbf{r} = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + \sin t\, \mathbf{k} \quad , \pi \xrightarrow{t} 0$$

och beräkna kurvintegralen enl.;

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\pi^0 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt =$$

$$= \int_\pi^0 \left(\sin^2 t\,(-\sin t) + \cos^2 t \sin t + \cos^2 t \sin t\right) dt =$$

$$= \int_\pi^0 (3\cos^2 t \sin t - \sin t)\, dt = \left[\cos t - \cos^3 t\right]_\pi^0 = \underline{\underline{0}}$$

<u>Uppg 5b</u> $\mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + 2u\mathbf{k}$

$$\iint_S y^2 z\, dS = \int_0^1 \left(\int_0^1 v^2 u \sqrt{1+4u^2}\, du \right) dv =$$

$$= \left[\frac{v^3}{3}\right]_0^1 \left[\frac{1}{12}(1+4u^2)^{3/2}\right]_0^1 = \underline{\underline{\frac{1}{36}(5^{3/2}-1)}}$$

<u>Uppg 6</u> a) $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi} \cdot \underbrace{\dfrac{\partial \xi}{\partial y}}_{=-1} + \dfrac{\partial u}{\partial \eta} \cdot \underbrace{\dfrac{\partial \eta}{\partial y}}_{=1} = \dfrac{\partial u}{\partial \eta} - \dfrac{\partial u}{\partial \xi}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial x}}_{=-1/x^2} + \frac{\partial u}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial x}}_{=0} = \frac{-1}{x^2} \frac{\partial u}{\partial \xi}$$

som insatt i diff.ekv. ger;

$$\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} = x^2 \left(\frac{-1}{x^2} \frac{\partial u}{\partial \xi} \right) \Leftrightarrow \frac{\partial u}{\partial \eta} = 0$$

b) $\frac{\partial u}{\partial \eta} = 0 \Rightarrow u = f(\xi)$, för godtycklig funktion f
(som är deriverbar)

Varje lösning till diff.ekv. har alltså formen;

$$\underline{\underline{u(x,y) = f\left(\tfrac{1}{x} - y\right)}}$$

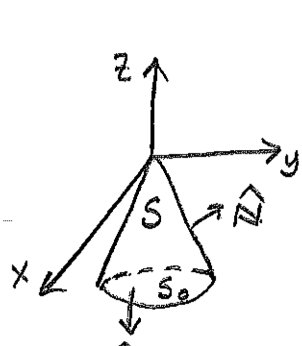
för någon deriverbar funktion f.

$u(1,y) = f(1-y) = y \Rightarrow f(y) = 1-y$

så den eftersökta lösningen är;

$$\underline{\underline{u(x,y) = 1 - \frac{1}{x} + y}}$$

Uppg 7 a) $\text{curl}\,\mathbb{F} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z+y^2 & x+z^2 & y+x^2 \end{vmatrix} = (1-2z)\,\mathbb{i} + (1-2x)\,\mathbb{j} + (1-2y)\,\mathbb{k}$

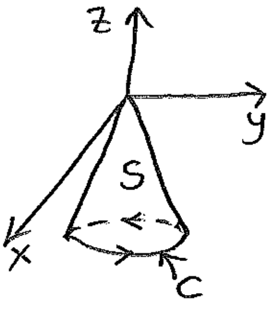


$$\iint_S \text{curl}\,\mathbb{F} \bullet \hat{\mathbb{N}}\,dS = \iiint_{\substack{x^2+y^2 \le z^2 \\ -2 \le z \le 0}} \underbrace{\text{div}(\text{curl}\,\mathbb{F})}_{=0}\,dV -$$

$$- \iint_{\left.\begin{matrix} x^2+y^2 \le 4 \\ z = -2 \end{matrix}\right\} S_0} \text{curl}\,\mathbb{F} \bullet \underset{\substack{\uparrow \\ -\mathbb{k} \text{ på } S_0}}{\hat{\mathbb{N}}}\,dS = \iint_{x^2+y^2 \le 4} (1-2y)\,dx\,dy = \iint_{x^2+y^2 \le 4} dx\,dy = \underline{\underline{4\pi}}$$

b) $\iint_S \text{curl}\,\mathbb{F} \bullet \hat{\mathbb{N}}\,dS = \oint_C \mathbb{F} \bullet d\mathbb{r} =$

$C \leftarrow \mathbb{r} = 2\cos t\,\mathbb{i} + 2\sin t\,\mathbb{j} - 2\mathbb{k},\ 0 \xrightarrow{t} 2\pi$



$$= \int_0^{2\pi} \left((-2+4\sin^2 t)(-2\sin t) + (2\cos t + 4)2\cos t + (2\sin t + 4\cos^2 t)\cdot 0 \right) dt$$

$$= 4\int_0^{2\pi} \cos^2 t\,dt = 4\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = 2\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \underline{\underline{4\pi}}$$