Error Estimates for FEM in 1D. —1—
two-point boundary value problem

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Lecture 5/1

1. The Dirichlet problem:

We study the equation

$$(BVP)\quad \begin{cases} -(a(x)u'(x))' = f(x), & 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

Assume that $a(x)>0$ is piecewise continuous in $(0,1)$, and $a(x)$ is bounded for $0\le x\le 1$.

Let $H_0^1 = \{v : \int_0^1 v'(x)^2\,dx < \infty,\ v(0)=v(1)=0\}$

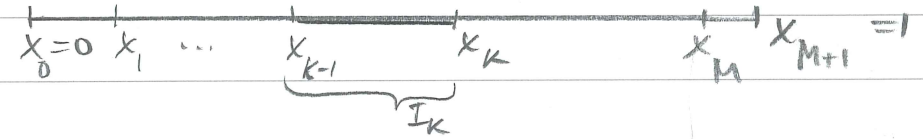
The variational formulation: Find $u\in H_0^1$ such that

$$(VF)\quad \int_0^1 a(x)u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \qquad \forall v\in H_0^1.$$

The Finite element method: Let

$T_h := \{0=x_0, x_1, x_2, \dots, x_M, x_{M+1}=1\}$ be a partition of $I=[0,1]$ into subintervals $I_k := [x_{k-1}, x_k]$, $k=1,\dots,M+1$ and set $h_k = x_k - x_{k-1}$.

We may define a piecewise constant function $h(x)=h_k$, for $x\in I_k$.



Let now

$$V_h^0 = \{v : v(x) \text{ is continuous, linear on each } I_k \text{ \& } v(0)=v(1)=0\}.$$

A finite element method for the (BVP) is: Find $u_h\in V_h^0$ such that

$$(FEM)\quad \int_0^1 a(x)u_h'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \qquad \forall v\in V_h^0.$$

If in (VF) we restrict v to $V_h^0\ (\subset H_0^1)$, then (VF)–(FEM)⇒

$$(G^\perp)\quad \int_0^1 a(x)\left(u'(x)-u_h'(x)\right)v'(x)\,dx = 0, \qquad \forall v\in V_h^0$$

This relation is called Galerkin orthogonality.

Error estimates for FEM in 1 D. -2-
Dirichlet BVP Problem

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Lecture 5/2

Measuring environment; We define the following norms:

$$\|v\|_a = \left(\int_0^1 a(x)\,|v(x)|^2\,dx\right)^{1/2} \qquad \text{(weighted } L_2\text{-norm)}$$

$$\|v\|_E = \left(\int_0^1 a(x)\,|v'(x)|^2\,dx\right)^{1/2} \qquad \text{(The Energy norm)}$$

$\|v\|_E$ describes the "elastic energy" for an elastic string modeled by our Dirichlet (BVP). Note that $\|v\|_E = \|v'\|_a$.

Error estimates in Energy norm

Theorem 1. Let u be a solution of Dirichlet BVP and u_h that of FEM. Then

$$\|u-u_h\|_E \le \|u-v\|_E, \quad \forall\ v \in \overset{\circ}{V}_h$$

Note. This means that the FE solution u_h is the best approximation of the solution u, in energy norm, in $\overset{\circ}{V}_h$.

Pf. $\|u-u_h\|_E^2 = \int_0^1 a\,(u'-u_h')^2\,dx = \int_0^1 a\,(u'-u_h')(u'-v'+v'-u_h')\,dx$

$$= \int_0^1 a\,(u'-u_h')(u'-v')\,dx + \int_0^1 a\,(u'-u_h')\underbrace{(v'-u_h')}_{\text{obs! } (v-u_h)\in \overset{\circ}{V}_h}\,dx \quad := I + II$$

By the $G^{\perp}$ $\quad II = 0$, hence

$$\|u-u_h\|_E^2 = \int_0^1 a(u'-u_h')(u'-v') = \int_0^1 a^{1/2}(u'-u_h')\cdot a^{1/2}(u'-v')\,dx \le \{C\text{-}S\}$$

$$\le \left(\int_0^1 a(x)\,(u'-u_h')^2\,dx\right)^{1/2} \cdot \left(\int_0^1 a(x)\,(u'-v')^2\,dx\right)^{1/2}$$

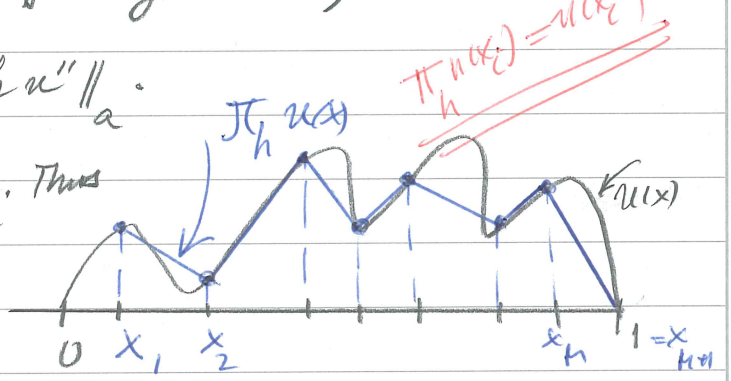
$$= \|u-u_h\|_E\,\|u-v\|_E \cdot \qquad \square$$

Next step is to show that there exists a function $v \in \overset{\circ}{V}_h$ such that $\|u-v\|_E$ is not "too-large". This is $\Pi_h u$: The piecewise linear interpolant of u.

Theorem 2 (An a priori error estimate), [Depends on the exact solution $u(x)$, which is unknown].

Let u and u_h be the solutions of the Dirichlet BVP and the FEM (defined above), respectively. Then, there is a constant C_1 (interpolation constant depending only on $a(x)$) such that

$$\|u-u_h\|_E \le C_1 \cdot \|h u''\|_a .$$



Pf. Note that $\Pi_h u \in \mathring{V}_h$. Thus by theorem 1:

$$\|u-u_h\|_E \le \|u-\Pi_h u\|_E = \|u'-(\Pi_h u)'\|_a = \left(\int_0^1 a(x)\left(u'(x)-(\Pi_h u)'(x)\right)^2 dx\right)^{1/2}$$

$$\le \max_x a(x)^{1/2} \underbrace{\left\{\int_0^1 \left(u'(x)-(\Pi_h u)'(x)\right)^2 dx\right\}^{1/2}}_{= \|u'-(\Pi_h u)'\|_{L_2} \le C_3 \|h u''\|_{L_2}}$$

$$\le C_3 \max_x a(x)^{1/2} \left(\int_0^1 \left(h(x)\, u''(x)\right)^2 dx\right)^{1/2}$$

$$\le C_3 \max_x a(x)^{1/2} \left(\int_0^1 \frac{a(x)}{\min_x a(x)} \left(h(x)\, u''(x)\right)^2 dx\right)^{1/2}$$

$$= \underbrace{C_3 \frac{\max_x \sqrt{a(x)}}{\min_x \sqrt{a(x)}}}_{:= C_1} \left(\int_0^1 a(x)\left(h(x)\, u''(x)\right)^2 dx\right)^{1/2} = C_1 \|h u''\|_a \cdot \square$$

Remark!
If the "objective" is to divide $(0,1)$ into a fixed, finite number of subintervals, then one can use Thm 2 to obtain an optimal partition, in the sense that: whenever $a(x)u''(x)$ gets large, we compensate by making $h(x)$ small. This, however, requires, of course, knowing the unknown solution $u(x)$.

Error Estimates for FEM in 1D -4-
Dirichlet problem [A posteriori]

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Lecture 5/4

We want to study "a posteriori" error analysis, where instead of the unknown solution u, we use the known values of the computed solution u_h.

Theorem 3. Let $e = u - u_h$ be the error of approximation and define the residual $R(u_h) = f(x) + (a(x)u_h'(x))'$.
$R(u_h(x))$ is well-defined except in the set $\{x_k\}_{k=0}^{M-1}$ of the partition.

A posteriori error Estimate: Under the assumption in thm 3:

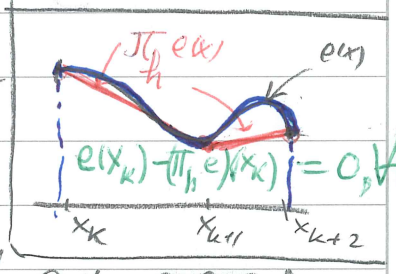
$$\|e\|_E \le C_i \left(\int_0^1 \frac{1}{a(x)} h^2(x) R(u_h(x))^2 dx \right)^{1/2}.$$

Pf. The error $e(x) = u(x) - u_h(x) \in H_0^1$ and thus

$$\|e\|_E^2 = \int_0^1 a(x) e'^2(x)\, dx = \int_0^1 a(x)\left(u'(x) - u_h'(x)\right) e'(x)\, dx$$

$$= \int_0^1 a(x) u'(x) e'(x)\, dx - \int_0^1 a(x) u_h'(x) e'(x)\, dx$$

$$\overset{(VF)}{=} \int_0^1 f(x) e(x)\, dx - \int_0^1 a(x) u_h'(x) e'(x)\, dx.$$



Now we add and subtract $\Pi_h e$, i.e. the interpolant of the error:

$$\|e\|_E^2 = \int_0^1 f(x)\left(e(x) - (\Pi_h e)(x)\right) dx + \underbrace{\int_0^1 f(x)(\Pi_h e)(x)\, dx}_{(FEM)=0} - \int_0^1 a(x) u_h'(x)\left(e'(x) - (\Pi_h e)'(x)\right) dx - \int_0^1 a(x) u_h'(x)(\Pi_h e)'(x)\, dx$$

Now since u_h is a solution of the (FEM) and $(\Pi_h e) \in V_h^0$, we have

$$(I)\quad \|e\|_E^2 = \int_0^1 f(x)\left(e(x) - \Pi_h e(x)\right) dx - \underbrace{\sum_{k=1}^{M+1} \int_{x_{k-1}}^{x_k} a(x) u_h'(x)\left(e'(x) - (\Pi_h e)'(x)\right) dx}_{= \int_0^1 a u_h'(x)\left(e' - (\Pi_h e)'\right) dx}$$

Integrating by parts over each subinterval (x_{k-1}, x_k) and using the fact that $e(x_k) - (\Pi_h e)(x_k) = 0$, $k = 0, 1, \dots, M+1$ $\Longrightarrow$

$$(II)\quad -\int_{x_{k-1}}^{x_k} a(x) u_h'(x)\left(e'(x) - (\Pi_h e)'(x)\right) dx = \int_{x_{k-1}}^{x_k} \left(a(x) u_h'(x)\right)'\left(e(x) - (\Pi_h e)(x)\right) dx + \underbrace{\left[\dots\left(e(x) - (\Pi_h e)(x)\right)\right]_{x_{k-1}}^{x_k}}_{=0} \Longrightarrow$$

Combining (I) & (II) we get

$$\|e\|_E^2 = \int_0^1 R(u_h(x))\,(e(x)-\pi_h e(x))\,dx = \int_0^1 \frac{1}{\sqrt{a(x)}}\,h(x)R(u_h(x))\cdot\sqrt{a(x)}\left(\frac{e(x)-\pi_h e(x)}{h(x)}\right)dx$$

$$\le\{C-S\}\le \left(\int_0^1 \frac{1}{a(x)}\,h^2(x)\,R(u_h(x))^2\,dx\right)^{1/2}\cdot \underbrace{\left\|\frac{e(x)-(\pi_h e)(x)}{h(x)}\right\|_a}_{:=\mathcal{F}}$$

$$\text{But } \mathcal{F} = \left(\int_0^1 a(x)\left(\frac{e(x)-\pi_h e(x)}{h(x)}\right)^2 dx\right)^{1/2} \le \max_x a(x)^{1/2}\left(\int_0^1\left(\frac{e(x)-\pi_h e(x)}{h(x)}\right)^2 dx\right)^{1/2}$$

$$\|e-\pi_h e\|_{\frac12}\le\|he'\|_{\frac12} \le C_1\max_x\sqrt{a(x)}\;\|e'\|_{\frac12}(0,1) = C_1\max_x\sqrt{a(x)}\left(\int_0^1 e'(x)^2\,dx\right)^{\frac12}$$

$$\le \underbrace{C_1\frac{\max\sqrt{a(x)}}{\min\sqrt{a(x)}}}_{\tilde C_1}\left(\int_0^1 a(x)\,e'(x)^2\,dx\right)^{\frac12} = \tilde C_1\|e'\|_a = \tilde C_2\|e\|_E \Rightarrow \text{pt. } \blacksquare$$

Remark 2 Now we have an error estimate which uses the approximate, i.e. computed solution and which can be used for mesh-refinements in the regions that is necessary. This is the ADAPTIVITY principle. The idea is:

Assume that one seeks an error bound

$$\|e\|_E \le TOL \quad \text{(error Tolerance)}.$$

(i) Make an initial partition of the interval.

(ii) Compute the corresponding FEM solution u_h and residual $R(u_h)$.

(iii) If $\|e\|_E > TOL$ refine the places for which $\frac{1}{a(x)}R(u_h)^2$ is large and perform (ii)–(iii) again.

