$$\begin{cases} \dot{u}(t) + a(t)\,u(t) = f(t), & 0 < t \le T \quad \text{(DE)} \\ u(0) = u_0 & \text{(IV)} \end{cases}$$

$a(t)$ bounded, $a(t) \ge 0 \Rightarrow$ parabolic problem

$a(t) > 0 \Rightarrow$ dissipative problem, $\dot{u} = \frac{du}{dt}$, $f(t)$ source term

The continuous solution:

Let $A(t) = \int_0^t a(r)\,dr$, then $u(t) = e^{-A(t)} u_0 + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds$.

Pf. Multiplying the (DE) by the integrations factor $e^{A(t)}$ yields;

$$\dot{u}(t) e^{A(t)} + \underbrace{\dot{A}(t)}_{=a(t)} e^{A(t)} u(t) = e^{A(t)} f(t) \qquad \left(\Leftrightarrow \frac{d}{dt}\left(u(t)e^{A(t)}\right) = e^{A(t)} f(t)\right).$$

Relabeling $t \rightsquigarrow s$ and integrating over $(0,t) \Rightarrow \int_0^t \frac{d}{ds}\left(u(s)e^{A(s)}\right) = \int_0^t e^{A(s)} f(s)\,ds$.

i.e. $u(t)e^{A(t)} - u(0)\underbrace{e^{A(0)}}_{=1} = \int_0^t e^{A(s)} f(s)\,ds \Rightarrow u(t) = e^{-A(t)} u_0 + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds$. $\boxtimes$

Stability

S1) $a(t) \ge \alpha > 0 \Rightarrow u(t) \le e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \le s \le t} |f(s)|$

S2) $a(t) \ge 0$, (i.e. $\alpha = 0$: parabolic case) $\Rightarrow |u(t)| \le |u_0| + \int_0^t |f(s)|\,ds$.

Pf. (S1) $a(t) \ge \alpha \Rightarrow \{\int_s^t du\} \Rightarrow \int_s^t a(u)\,du \ge \alpha \int_s^t du \Rightarrow \underline{A(t) - A(s) \ge \alpha(t-s)}$ $(*)$

Then $e^{-A(t)} \le e^{-\alpha t}$ (Let $s=0$ in $(*)$) I and $e^{-(A(t)-A(s))} \le e^{-\alpha(t-s)}$ II

Inserting in exact solution $\Rightarrow |u(t)| \le e^{-\alpha t}|u_0| + \int_0^t e^{-\alpha(t-s)}|f(s)|\,ds$

i.e. $|u(t)| \le e^{-\alpha t}|u_0| + \max_{0 \le s \le t}|f(s)| \int_0^t e^{-\alpha(t-s)}\,ds = e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \le s \le t}|f(s)|$ $\boxtimes$

$\int_0^t e^{-\alpha(t-s)}\,ds = \frac{1}{\alpha} e^{-\alpha(t-s)}\Big|_{s=0}^{t}$

(S2) Let $\alpha = 0$ in I & II then by the continuous solution

$$|u(t)| \le |u_0| + \int_0^t |f(s)|\,ds \ \boxtimes$$

Remark: Note that in S1) the initial solution is damped by increasing t & only the maximum of the source term f matters, whereas in S2) the solution is dominated by non-damped initial data and integral of the source term (which dominates).

Continuous Galerkin of degree 1; cG(1): Rely on piecewise linear, continuous trial functions + piecewise constant discontinuous test functions.

Discontinuous Galerkin of degree 0, dG(0):

Piecewise constant, discontinuous trial and test functions.

Global Galerkin of degree q: find $U\in P^q(0,T)$, with $U(0)=u_0$ such that

$$\int_0^T (\dot{U}+aU)v\,dt = \int_0^T f v\,dt, \quad \forall v\in P^q(0,T), \quad v(0)=0.$$

$V:=\{t, t^2, \dots, t^q\}$.

Continuous Galerkin of degree q: find $U\in P^q(0,T)$, with $U(0)=u_0$ such that

$$\int_0^T (\dot{U}+aU)v\,dt = \int_0^T f v\,dt, \quad \forall v\in P^{q-1}(0,T);$$

$V: \{1, t, \dots, t^{q-1}\}$.

Here $q=1 \Rightarrow v\equiv 1 \Rightarrow U(T)-U(0)+a\int_0^T\left(U(T)\frac{t}{T}+U(0)\frac{T-t}{T}\right)dt=\int_0^T f\,dt$

i.e. $U(T)-U(0)+aU(T)\left(\frac{t}{T}\right)\Big|_0^T - aU_0\left\{\frac{(T-t)^2}{T}\right\}\Big|_0^T = \int_0^T f\,dt \Leftrightarrow$

$U(T)-U_0+aU(T)+aU_0T=\int_0^T f\,dt \Leftrightarrow U(T)\overset{*}{=}\frac{(1-aT)}{1+aT}U_0+\frac{1}{1+aT}\int_0^T f\,dt.$

$U(t)$: linear & $U(0)=u_0 \overset{*}{\Rightarrow} U(t),\ t\in[0,T]$.

Algorithm for the cG(1) on a partition T_h; $I_k=[t_{k-1}, t_k]$, of $[0,T]$:

I. Compute $U(t_1)$ applying $(*)$ to $(0, t_1)$ with $U(0)=u_0$.

II_a. Assume that U is computed on $[t_{n-2}, t_{n-1}]$

II_b. Compute $U(t_n)$ using $(*)_n$ as follows: $\int_{t_{n-1}}^{t_n}(\dot{U}+aU)dt=\int_{t_{n-1}}^{t_n} f\,dt$

$\Leftrightarrow U(t_n)-U(t_{n-1})+\int_{t_{n-1}}^{t_n} a\left(U(t_{n-1})\frac{t_n-t}{t_n-t_{n-1}}+U(t_n)\frac{t-t_{n-1}}{t_n-t_{n-1}}\right)dt=\int_{t_{n-1}}^{t_n} f\,dt$

$\Rightarrow U(t_n)$ & since $U(t_{n-1})$ is known & U is linear we get $U(t)$ for $t\in[t_{n-1}, t_n]$.

Global forms: $\tau_k := \{0=t_0<\dots<t_N=T\}$ -3-
is a partition of the time interval $[0,T]$.

M.A.
Lecture 7/3

Continuous Galerkin cG(q): find $U\in V_k^{(q)}$ such that $U(0)=u_0$ &

$$\int_0^{t_N}(\dot U + aU)\,v\,dt = \int_0^{t_N} f v\,dt \quad \forall v\in W_k^{(q-1)}.$$

$V_k^{(q)}=\{v:\ v$ continuous piecewise polynomial of degree $\le q$ on $\mathcal{T}_k\}$

$W_k^{(q-1)} = \{w:\ w$ discontinuous piecewise polynomial of degree $\le q-1$ on $\mathcal{T}_k\}$.

Discontinuous Galerkin dG(q): find $U\in P^q(0,T)$ such that

$$\int_0^T(\dot U+aU)\,v\,dt = \alpha\big(U(0)-u(0)\big)v(0) = \int_0^T f v\,dt \quad \forall v\in P^q(0,T)$$

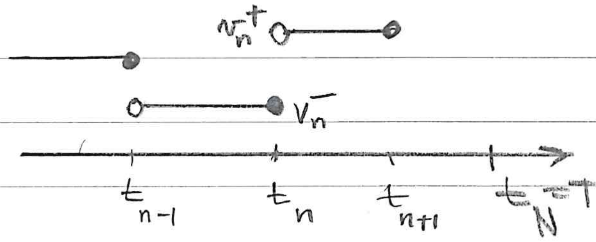
(Note: Here $P^q(0,T)\equiv W_k^{(q)}$ & dG(q) gives up the requirement that u satisfies the initial condition).

$\alpha=1 \Rightarrow$ Best accuracy and stability.

Notation: In the sequel we use $U_n = U(t_n)$, $U_{n-1}=U(t_{n-1})$, ... &

Define $v_n^{\pm} = \lim_{s\to 0} v(t_n\pm s)$,

$[v_n] = v_n^+ - v_n^-$.



dG(q): for $n=1,\dots,N$ find $U\in P^q(t_{n-1},t_n]$ such that

$$\int_{t_{n-1}}^{t_n}(\dot U+aU)\,v\,dt + U_{n-1}^+ v_{n-1}^+ = \int_{t_{n-1}}^{t_n} f v\,dt + U_{n-1}^- v_{n-1}^+, \quad \forall v\in P^q(I_n).$$

For $q=0$, (approx. with piecewise constants); $v\equiv 1$, $U=U_n=U_{n-1}^+=U_n^-$ on I_n

$\rightarrow$ dG(0): for $n=1,\dots,N$ find piecewise constant U_n such that

$$\int_{t_{n-1}}^{t_n} aU_n\,dt + U_n = \int_{t_{n-1}}^{t_n} f\,dt + U_{n-1}.$$

(OBS! Here $\dot U\equiv 0$, $v\equiv 1$.)

Finally summing over n in dG(q) we get global dG(q): find $U\in W_k^{(q)}$:

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(\dot U+aU)\,v\,dt + \sum_{n=1}^{N}[U_{n-1}]\,v_{n-1}^+ = \int_0^{t_N} f v\,dt, \quad \forall v\in W_k^{(q)};\ U_0^-=u_0.$$

Problem: $\dot{u}(t)+a(t)u(t)=f(t), \quad \forall t\in(0,T)$ (DE) (*)

Variational form: $\displaystyle\int_0^T(\dot{u}+au)v\,dt=\int_0^T fv\,dt, \ \forall v \overset{PI}{\Longrightarrow}$

$$u(T)v(T)-u(0)v(0)+\int_0^T u(-\dot{v}+av)\,dt=\int_0^T fv\,dt \qquad (**)$$

If we choose v to be the solution of the dual problem

$-\dot{v}+av=0$, in $(0,T)$ Then $(**)\Leftrightarrow$

$$u(T)v(T)=u(0)v(0)+\int_0^T fv\,dt.$$

The dual problem for (*): find $\varphi(t)$ such that

$$-\dot{\varphi}(t)+a(t)\varphi(t)=0 \quad \text{for} \quad t_N>t\geq 0$$

$$\varphi(t_N)=e_N, \quad e_N=u_N-U_N=u(t_N)-U(t_N).$$

Thm For $N=1,2,\dots$, the cG(1) solution U, for (*), satisfies

I $\quad |e_N|\leq S(t_N)\cdot\max_{[0,t_N]}|kr(U)|,$

where $k=k_n=|I_n|$ for $t\in I_n=(t_{n-1},t_n)$, $\underline{r(U)=\dot{U}+aU-f}$: The residual error

Further the stability factor

II $\quad S(t_N)=\dfrac{\int_0^{t_N}|\dot{\varphi}|\,dt}{|e_N|}\leq\begin{cases} e^{\lambda t_N} & \text{if } |a(t)|\leq\lambda, \ \forall t\\ 1 & \text{if } a(t)\geq 0, \ \forall t.\end{cases}$

II. $-\dot{\varphi}+a\varphi=0 \Longrightarrow e_N^2=e_N^2+0=e_N^2+\displaystyle\int_0^{t_N}e(-\dot{\varphi}+a\varphi)\,dt=[PI]$

$=e_N^2-e\varphi\Big|_0^{t_N}+\displaystyle\int_0^{t_N}(\dot{e}\varphi+ae\varphi)\,dt=\{e(0)=0\}=\cancel{e_N^2}-\cancel{e(t_N)\underbrace{\varphi(t_N)}_{=e_N}}+\int_0^{t_N}(\dot{e}+ae)\varphi\,dt$

$=\{\dot{e}+ae=\dot{u}-\dot{U}+au-aU=f-\dot{U}-aU=-r(U)\}=-\displaystyle\int_0^{t_N}r(U)\varphi\,dt.$

(The error representation formula)

Now let $\Pi_k\varphi=\dfrac{1}{k_n}\displaystyle\int_{I_n}\varphi\,ds$, then

$$e_N^2=-\int_0^{t_N}r(U)(\varphi-\Pi_k\varphi)\,dt+\underbrace{\int_0^{t_N}r(U)\Pi_k\varphi\,dt}_{=0\ (\text{FEM})}.$$

Now by mean value theorem, $\exists \xi \in I_n$ s.t. $\frac{1}{k_n}\int_{I_n} \varphi\, ds = \varphi(\xi)$.

Thus $\int_{I_n} |\varphi - \pi_k\varphi|\,dt = \int_{I_n} |\varphi(t) - \varphi(\xi)|\,dt = \int_{I_n} \left| \int_{\xi}^{t} \dot\varphi(s)\,ds \right| dt$ (***)

$$\leq k_n \int_{I_n} |\dot\varphi(s)|\,ds.$$

Let $|v|_J = \max_{t\in I_J} |v(t)|$ & set $\int_0^{t_N} |\varphi - \pi_k\varphi|\,dt = \sum_{n=1}^{N}\int_{I_n} \dots$, Then

$$|e_N| \leq \sum_{n=1}^{N}\left(|r(U)|_{I_n} \int_{I_n} |\varphi - \pi_k\varphi|\,dt\right) \leq \sum_{n=1}^{N} |r(U)|_{I_n}\, k_n \int_{I_n} |\dot\varphi|\,dt$$

$$\leq \max_{1\leq n\leq N}\left(k_n |r(U)|_{I_n}\right) \int_0^{t_N} |\dot\varphi|\,dt \leq \cancel{|e_N|}\, S(t_N) \max_{[0,t_N]} |kr(U)|$$

$\frac{\int_0^{t_N} |\dot\varphi|}{|e_N|}$ ∎

To prove (II); we transform the dual problem to a forward problem, by a change of variables: $s = t_N - t$, ($t = t_N - s$).
Then, using the chain rule $\psi(s) := \varphi(t_N - s)$, satisfies

$$\underset{=\varphi(t_N)}{}$$

$$\frac{d\psi}{ds} = \frac{d\varphi}{dt}\cdot\frac{dt}{ds} = -\dot\varphi(t_N - s),$$ and the dual problem reads as

$$-\dot\varphi(t_N - s) + a(t_N - s)\,\varphi(t_N - s) = 0.$$

The corresponding forward problem is:
$$\begin{cases} \dfrac{d\psi(s)}{ds} + a(t_N - s)\,\psi(s) = 0, & 0 \leq s \leq t_N \\ \psi(0) = \varphi(t_N) = e_N . \end{cases}$$

Using the solution formula (with $u_0 = e_N$ & $f \equiv 0$)

$$\begin{cases} \psi(s) = \psi(0)\, e^{-\int_0^s a(t_N - r)\,dr} \underset{\left\{\substack{t_N - r = v \\ -dr = dv}\right\}}{=} e_N\, e^{\int_{t_N}^{t_N - s} a(v)\,dv} \\ = e_N\, e^{A(t_N - s) - A(t_N)} \end{cases}$$

$$\Rightarrow \varphi(t) = e_N\, e^{A(t) - A(t_N)} \quad \& \quad \dot\varphi(t) = e_N\, a(t)\, e^{A(t) - A(t_N)}$$

$$\Rightarrow \int_0^{t_N} |\dot\varphi(t)|\,dt = |e_N| \int_0^{t_N} a(t)\, e^{A(t) - A(t_N)}\,dt = |e_N| \left[e^{A(t) - A(t_N)} \right]_{t=0}^{t=t_N}$$

$$= |e_N| \left(1 - e^{-A(t_N)}\right) \leq |e_N|$$

$$\Rightarrow S(t_N) \leq 1. \; \blacksquare$$

The case $|a(t)| \leq \lambda \Rightarrow |\dot{\varphi}(t)| \leq \lambda e^{A(t)-A(t_N)} |e_N|$.

$$\Rightarrow |\dot{\varphi}(t)| \leq \lambda e^{\int_{t_N}^{t} a(s)\,ds} |e_N| \leq \lambda e^{\lambda(t_N-t)} |e_N|$$

$$\Rightarrow \int_0^{t_N} |\dot{\varphi}(t)|\,dt \leq |e_N| \int_0^{t_N} \lambda e^{\lambda(t_N-t)}\,dt = |e_N|\left(-e^{\lambda(t_N-t)}\Big|_0^{t_N}\right)$$

$$= |e_N|(-1+e^{\lambda t_N}) \Rightarrow S(t_N) \leq -1+e^{\lambda t_N}. \quad \boxtimes$$

<u>Convergence of order $O(k^2)$:</u>

Note that $(g-\Pi_k g) \perp$ Constant $\forall g$

Because $\Pi_k g = \frac{1}{k_n}\int_{I_n} g \Rightarrow \int_{I_n} (g-\Pi_k g)\cdot c = \int_{I_n}\left(g - \frac{1}{k_n}\int_{I_n} g\right)\cdot c$

$$= \left(\int_{I_n} g - \frac{1}{k_n}\cdot k_n\left(\int_{I_n} g\right)\right)\cdot c = \left(\int_{I_n} g - \int_{I_n} g\right)\cdot c = 0\cdot c = 0.$$

Now since $\dot{U}$ is constant on I_n we use error representation formula to write:

$$e_N^2 = -\int_0^{t_N} r(U)(\varphi - \Pi_k\varphi)\,dt = \int_0^{t_N} (f - aU - \underbrace{\dot{U})(\varphi-\Pi_k\varphi)}_{=0}\,dt$$

$$= -\int_0^{t_N} \underbrace{\left((aU-f) - \Pi_k(aU-f)\right)(\varphi - \Pi_k\varphi)}_{=0}\,dt \Rightarrow$$

$$\text{Thm} \Rightarrow |e_N| \leq S(t_N)\left|k\,\underbrace{|(aU-f)-\Pi_k(aU-f)|}\right|_{[0,t_N]} \underset{\text{error}}{\overset{\text{Interpol}}{\leq}} k\left|\frac{d}{dt}(aU-f)\right|_{[0,t_N]}$$

$$\Rightarrow |e_N| \leq S(t_N)\left|k^2\frac{d}{dt}(aU-f)\right|_{[0,t_N]}. \quad \boxtimes$$

Remark: But it not! so practical to differentiate the residual $(aU-f)$.