Initial Boundary Value Problem (IBVP)

The Heat Equation in 1-spatial dimen -1-

IIA.

Lecture 9/-1-

(PDE) $\dot{u} - u'' = f, \quad 0<x<1, \quad t>0$

I.C. $u(x,0) = u_0(x), \quad 0<x<1,$

B.C. $u(0,t) = u_x(1,t) = 0, \quad t>0$ (IBVP)

Theorem The (IBVP) satisfies the following stability estimates.

(a) $\|u(\cdot,t)\| \le \|u_0\| + \int_0^t \|f(\cdot,s)\|\, ds,$

(b) $\|u_x(\cdot,t)\|^2 \le \|u_0'\|^2 + \int_0^t \|f(\cdot,s)\|^2 ds.$

$\|w(\cdot,t)\| := \left(\int_0^1 w^2(x,t)\,dx\right)^{1/2}$

Pf. (a): Multiply the PDE by u and integrate over $x\in(0,1) \Rightarrow$

$$\int_0^1 \dot{u}u\,dx - \int_0^1 u''u\,dx = \int_0^1 fu\,dx. \iff \{\text{PI in } 2^{\text{nd}} \text{ term}\}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \int_0^1 u'^2\,dx - \Big[\underbrace{u'(x,t)}_{\to 0}\, u(x,t)\Big]_{x=0}^{x=1} = \int_0^1 fu\,dx. \qquad (A)$$

Thus $\|u\|\frac{d}{dt}\|u\| + \|u'\|^2 = \int_0^1 fu\,dx \le \{C\text{-}S\} \le \|f\|\,\|u\|$

$\Rightarrow \{\|u'\|^2 \ge 0\} \Rightarrow \|u\|\frac{d}{dt}(\|u\|) \le \|f\|\|u\| \Rightarrow \frac{d}{dt}(\|u\|) \le \|f\|.$

$\int_0^t \ldots ds \Rightarrow$ $\int_0^t \frac{d}{ds}\|u(\cdot,s)\|\,ds \le \int_0^t \|f(\cdot,s)\|\,ds \Rightarrow$

$$\|u(\cdot,t)\| - \underbrace{\|u(\cdot,0)\|}_{=u_0} \le \int_0^t \|f(\cdot,s)\|\,ds \Rightarrow a)$$

(b): Multiply the Equation by $\dot{u}$ & integrate over $x\in(0,1) \Rightarrow$

$$\int_0^1 (\dot{u})^2\,dx - \int_0^1 u''\dot{u}\,dx = \int_0^1 f\dot{u}\,dx \xrightarrow[\text{in } 2^{\text{ed}} \text{ term}]{\text{PI}}$$

$$\|\dot{u}\|^2 + \int_0^1 u'\dot{u}'\,dx - \Big[\underbrace{u'(x,t)}_{\to 0}\,\dot{u}(x,t)\Big]_{x=0}^{x=1} = \int_0^1 f\dot{u} \le \{C\text{-}S\} \le \|f\|\,\|\dot{u}\| \Rightarrow$$

$$\|\dot{u}\| + \frac{1}{2}\frac{d}{dt}\|u'\|^2 \le \frac{1}{2}\|f\|^2 + \frac{1}{2}\|\dot{u}\|^2 \Rightarrow \|\dot{u}\|^2 + \frac{d}{dt}\|u'\|^2 \le \|f\|^2$$

$$\Rightarrow \frac{d}{dt}\|u'\|^2 \le \|f\|^2 \xrightarrow{\int_0^t \ldots ds} \int_0^t \frac{d}{ds}\|u'(\cdot,s)\|^2\,ds \le \int_0^t \|f(\cdot,s)\|^2 ds$$

$$\Rightarrow \|u'(\cdot,t)\|^2 - \|u'(\cdot,0)\|^2 \le \int_0^t \|f(\cdot,s)\|^2\,ds \Rightarrow b) \quad \square$$

Thm 2 For the homogeneous case we have that

(a) $\frac{d}{dt}\|u\|^2 + 2\|u'\|^2 = 0,$

(b) $\|u(\cdot,t)\| \le e^{-t}\|u_0\|$.

Pf. (a): Thm 1 (a) $\Rightarrow$ (A) $\overset{f\equiv 0}{\Rightarrow}$ $\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u'\|^2 = 0 \Rightarrow$ (a).

(b): (a) Together with poincaré inequality: $\|u\| \le \|u'\| \Rightarrow$

$$\frac{d}{dt}\|u\|^2 + 2\|u\|^2 \le 0 \quad (\cdot e^{2t} \text{ (I.F.)})$$

$$\frac{d}{dt}\left(\|u\|^2 e^{2t}\right) = \left(\frac{d}{dt}\|u\|^2 + 2\|u\|^2\right)e^{2t} \le 0$$

$$\Rightarrow \int_0^t \frac{d}{ds}\left(\|u(\cdot,s)\|^2 e^{2s}\right) ds \le 0$$

$$\Leftrightarrow \|u(\cdot,t)\|^2 e^{2t} - \|u(\cdot,0)\|^2 \le 0$$

$$\Leftrightarrow \|u(\cdot,t)\|^2 \le e^{-2t}\|u(\cdot,0)\|^2$$

$$\Leftrightarrow \|u(\cdot,t)\| \le e^{-t}\|u(\cdot,0)\| \Rightarrow b) \blacksquare$$

Thm 3 Consider the homogeneous (IBVP) above, For any small $\varepsilon > 0$,

$$\int_\varepsilon^t \|\dot{u}\|(s)\, ds \le \frac{1}{2}\sqrt{\ln\frac{t}{\varepsilon}}\,\|u_0\|.$$

Pf Multip the (PDE) by $-tu''$ & $\int_0^1 \cdots dx \Rightarrow$ $-t\int_0^1 \dot{u}u'' + t\int_0^1 (u')^2 = 0$ (B)

[PI] $\Rightarrow -\int_0^1 \dot{u}u'' = \int_0^1 \dot{u}'u'\,dx - \left[\dot{u}(x,t)\,u'(x,t)\right]_{x=0}^{x=1} = \int_0^1 \dot{u}'u' = \frac{1}{2}\frac{d}{dt}\|u'\|^2.$

Inserting in (B) $\Rightarrow$ $t\frac{1}{2}\frac{d}{dt}\|u'\|^2 + t\|u''\|^2 = 0 \Rightarrow \{t\frac{d}{dt}\|u'\|^2 = \frac{d}{dt}\left(t\|u'\|^2\right) - \|u'\|^2$

$\Rightarrow \frac{d}{dt}\left(t\|u'\|^2\right) + 2t\|u''\|^2 = \|u'\|^2 \Rightarrow \{t\,\cdots\,s\,\int_0^t \cdots ds\} \Rightarrow$

(C) $\int_0^t \frac{d}{ds}\left(s\|u(\cdot,s)\|^2\right)ds + 2\int_0^t s\|u'(\cdot,s)\|^2 ds \le \int_0^t \|\dot{u}(\cdot,s)\|^2 ds \le \frac{1}{2}\|u_0\|^2.$ (?)

To show (?) $\int_0^t \dots ds$ in Thm 2 (a) $\Rightarrow \int_0^t \frac{d}{ds}\|u(\cdot,s)\|^2 + 2\int_0^t \|u'(\cdot,s)\|^2 = 0$

$$\Rightarrow \|u(\cdot,t)\|^2 - \|u(\cdot,0)\|^2 + 2\int_0^t \|u'(\cdot,s)\|^2 ds = 0$$

$$\Rightarrow \int_0^t \|u'(\cdot,s)\|^2 ds = -\tfrac{1}{2}\|u(\cdot,s)\|^2 + \tfrac{1}{2}\|u_0\|^2 \le \tfrac{1}{2}\|u_0\|^2 \Rightarrow (?).$$

Thus (C) can be rewritten as

$$t\|u'(\cdot,t)\|^2 + 2\int_0^t s\|u''(\cdot,s)\|^2 ds \le \tfrac{1}{2}\|u_0\|^2$$

In particular we have

(I) $\|u'(\cdot,t)\| \le \frac{1}{\sqrt{2t}}\|u_0\|$ and (II) $\left(\int_0^t s\|u''(\cdot,s)\|^2 ds\right)^{1/2} \le \frac{1}{2}\|u_0\|$.

(Compare with Thm 2 (b) $\|u(\cdot,t)\| \le e^{-t}\|u_0\|$)

Now by the PDE $\dot{u} = u''$, Thus

$$\int_\varepsilon^t \|\dot{u}(\cdot,s)\| ds = \int_\varepsilon^t \|u''(\cdot,s)\| ds = \int_\varepsilon^t 1\cdot\|u''(\cdot,s)\| ds$$

$$= \int_\varepsilon^t \frac{1}{\sqrt{s}}\cdot\sqrt{s}\,\|u''(\cdot,s)\| ds \le \{C\text{-}S\}$$

$$\le \left(\int_\varepsilon^t s^{-1} ds\right)^{1/2}\cdot\left(\int_\varepsilon^t s\|u''\|^2 ds\right)^{1/2} \overset{(II)}{\le} \frac{1}{2}\sqrt{\ln\tfrac{t}{\varepsilon}}\,\|u_0\|. \quad \square$$

To continue show that (OBS! your assignment!)

(III) $\|u''(\cdot,t)\|$ $\left(= \|\dot{u}(\cdot,t)\|\right) \le \frac{1}{\sqrt{2}\,t}\|u_0\|$.

Hint: Multip (PDE) by $t^2(u'')$, (or diff $\dot{u}-u''$ w.r.t. t and multip the result by $t^2\dot{u}$) ...

Remark: Compare (I) & (III) with Thm 1 (a) & Thm 2 (b): $\{\|u\| \le \|u_0\|\}$ $\{\|u\| \le e^{-t}\|u_0\|\}$. (with $f \equiv 0$)

BVP Consider the one dimensional heat equation

$$\begin{cases} \dot{u} - u'' = f & 0<x<1, \quad t>0 \quad (PDE) \\ u(0,t) = u(1,t) = 0 & t>0 \quad (\text{Dirichlet B.C.}) \\ u(x,0) = u_0(x), & 0<x<1 \quad (\text{initial data}) \end{cases}$$

VF: Variational formulation : for every time interval $I_n = (t_{n-1}, t_n)$

find u such that

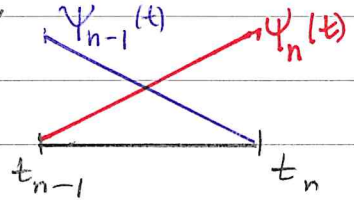
$$(VF) \quad \int_{I_n} \int_0^1 (\dot{u} v + u' v') \, dx \, dt = \int_{I_n} \int_0^1 f v \, dx \, dt$$

for all v such that $v(0,t) = v(1,t) = 0$.

FEM : A piecewise linear Galerkin approximation :

For each time interval $I_n = (t_{n-1}, t_n)$ let

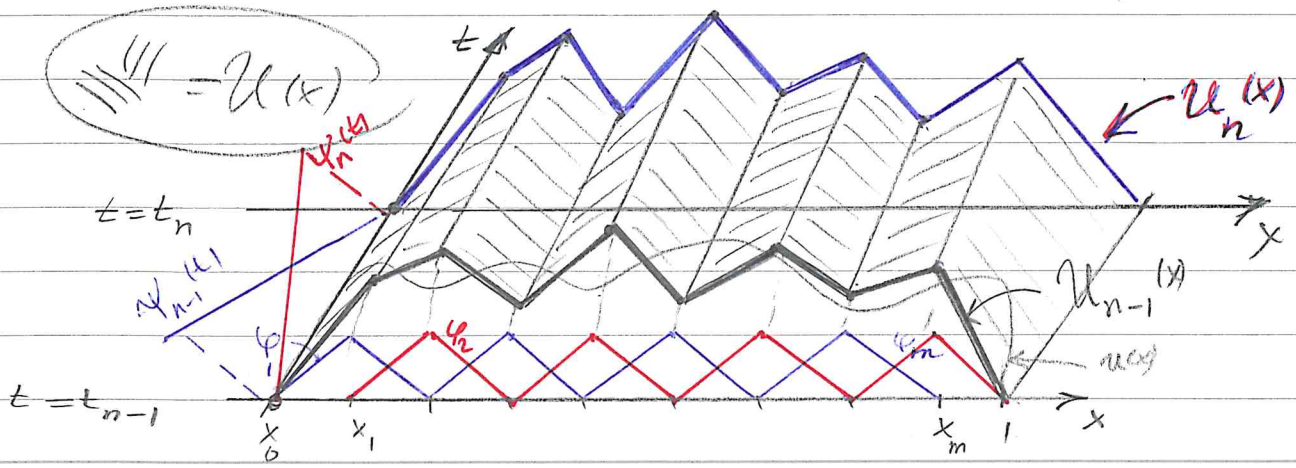
$$(FEM) \quad U(x,t) = U_{n-1}(x) \psi_{n-1}(t) + U_n(x) \psi_n(t),$$



where $\psi_{n-1}(t) = \frac{t_n - t}{k}$, $\psi_n = \frac{t - t_{n-1}}{k}$

$$U_{\tilde{n}}(x) = U_{\tilde{n},1} \varphi_1(x) + U_{\tilde{n},2} \varphi_2(x) + \cdots + U_{\tilde{n},m} \varphi_m(x), \quad \tilde{n} = n-1, n.$$

i.e. U is piecewise linear (here continuous) both in space & time.



Here the unknowns are $U_{n,k}$, $k=1,2,\dots,m$, ($U_{n-1,k}$ assumed are given).

Discrete variational formulation

$$\text{(DVF)}\quad \int_{I_n}\int_0^1 \left(\dot U \varphi_i + U' \varphi_i'\right) dx\,dt = \int_{I_n}\int_0^1 f\varphi_i(x)\,dx\,dt,\quad i=1,2,\dots,m.$$

In our example $\dot U = \dfrac{U_n - U_{n-1}}{k}$, $U' = U_{n-1}(t_{n-1})\psi_{n-1}'(t) + U_n(x)\psi_n'(t)$

Thus

$$\int_{I_n}\int_0^1 \dot U \varphi_i\,dx\,dt = \frac{1}{k}\int_{I_n}\Big(\int_0^1 \underbrace{(U_n - U_{n-1})\varphi_i(x)\,dx}_{\uparrow \text{independent of } t}\,dt$$

$$= \int_0^1 U_n(x)\varphi_i(x)\,dx - \int_0^1 U_{n-1}(x)\varphi_i(x)\,dx$$

Inserting in (DVF) $\Rightarrow \{U_n(x) = \sum_{k=1}^m U_{n,k}\varphi_j(x),\ j=1,\dots,m\} \Rightarrow$

$$\underbrace{\int_0^1 U_n\varphi_i(x)}_{MU_n} - \underbrace{\int_0^1 U_{n-1}\varphi_i(x)}_{MU_{n-1}} + \underbrace{\int_{I_n}\psi_{n-1}(t)}_{\frac{k}{2}}\underbrace{\int_0^1 U_{n-1}'(x)\varphi_i'(x)}_{SU_{n-1}} + \underbrace{\int_{I_n}\psi_n(t)}_{\frac{k}{2}}\underbrace{\int_0^1 U_n'(x)\varphi_i'(x)}_{SU_n}$$

$$= \int_{I_n}\int_0^1 f\varphi_i\,dx\,dt,$$

which can be written in a compact form as the Crank-Nicolson system, viz.

$$\boxed{\left(M + \tfrac{k}{2}S\right)U_n = \left(M - \tfrac{k}{2}\right)U_{n-1} + F_n} \quad \text{(CNS)}$$

$\dots\Rightarrow\dots$ $U_n = [U_{n,1}, U_{n,2}, \dots, U_{n,m}]^T$.

Thus for each $n=1,\dots,N$, given U_{n-1} and F_n (i.e., f), we can find U_n using (CNS).

Note that in $S\mathcal{U}_n = \int_0^1 \mathcal{U}_n' \varphi_i'$, $i=1,\dots,m$,

S is the stiffness matrix:

$$\mathcal{U}_n(x) = \mathcal{U}_{n,1}\varphi_1(x) + \mathcal{U}_{n,2}\varphi_2(x) + \dots + \mathcal{U}_{n,m}\varphi_m(x) \Rightarrow$$

$$\mathcal{U}_n'(x) = \mathcal{U}_{n,1}\varphi_1'(x) + \mathcal{U}_{n,2}\varphi_2'(x) + \dots + \mathcal{U}_{n,m}\varphi_m'(x),$$

Hence,

$$S\mathcal{U}_n = \int_0^1 \mathcal{U}_n'(x)\varphi_i'(x) = \left(\int_0^1 \varphi_i'(x)\varphi_1'(x)\right)\mathcal{U}_{n,1} + \left(\int_0^1 \varphi_i'(x)\varphi_2'(x)\right)\mathcal{U}_{n,2} + \dots$$

$$+ \dots + \left(\int_0^1 \varphi_i'(x)\varphi_m'(x)\right)\mathcal{U}_{n,m}$$

$$= \sum_{j=1}^{m} \left(\int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx\right)\mathcal{U}_{n,j}$$

which in matrix form is:

$$S\mathcal{U}_n = \begin{pmatrix} \int_0^1 \varphi_1'\varphi_1' & \int_0^1 \varphi_1'\varphi_2' & \cdots & \int_0^1 \varphi_1'\varphi_m' \\ \int_0^1 \varphi_2'\varphi_1' & \int_0^1 \varphi_2'\varphi_2' & \cdots & \int_0^1 \varphi_2'\varphi_m' \\ \vdots & & & \\ \int_0^1 \varphi_m'\varphi_1' & \int_0^1 \varphi_m'\varphi_2' & \cdots & \int_0^1 \varphi_m'\varphi_m' \end{pmatrix} = \overbrace{\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & \vdots & & & \\ & & 0 & -1 & 2 & -1 \\ 0 & & \cdots & 0 & -1 & 2 \end{bmatrix}}^{S} \begin{bmatrix} \mathcal{U}_{n,1} \\ \mathcal{U}_{n,2} \\ \vdots \\ \mathcal{U}_{n,m} \end{bmatrix}$$

Likewise

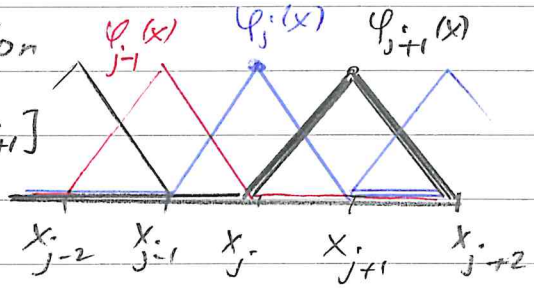
$$M\mathcal{U}_n = \int_0^1 \mathcal{U}_n\varphi_i = \underbrace{\begin{pmatrix} \int_0^1 \varphi_1\varphi_1 & \int_0^1 \varphi_1\varphi_2 & \cdots & \int_0^1 \varphi_1\varphi_m \\ \int_0^1 \varphi_2\varphi_1 & \int_0^1 \varphi_2\varphi_2 & \cdots & \int_0^1 \varphi_2\varphi_m \\ \vdots & & & \\ \int_0^1 \varphi_m\varphi_1 & \int_0^1 \varphi_m\varphi_2 & \cdots & \int_0^1 \varphi_m\varphi_m \end{pmatrix}}_{\text{Mass-matrix}} \mathcal{U}_n$$

Next we compute the

Mass-matrix (OBS! Uniform partition)

Recall that, for a uniform partition

$$\varphi_j(x) = \frac{1}{h}\begin{cases} 0, & x \notin [x_{j-1}, x_{j+1}] \\ x - x_{j-1}, & x_{j-1} \le x \le x_j \\ x_{j+1} - x, & x_j \le x \le x_{j+1} \end{cases}$$



$$\Rightarrow \int_0^1 \varphi_j^2(x)\,dx = \frac{1}{h^2}\left(\int_{x_{j-1}}^{x_j}(x-x_{j-1})^2\,dx + \int_{x_j}^{x_{j+1}}(x_{j+1}-x)^2\,dx\right)$$

$$= \frac{1}{h^2}\left(\left[\frac{(x-x_{j-1})^3}{3}\right]_{x_{j-1}}^{x_j} + \left[\frac{(x_{j+1}-x)^3}{-3}\right]_{x_j}^{x_{j+1}}\right)$$

$$= \frac{1}{h^2}\left(\frac{h^3}{3} + \frac{h^3}{3}\right) = \frac{2}{3}h$$

$$\int_0^1 \varphi_j(x)\,\varphi_{j+1}(x)\,dx = \frac{1}{h^2}\int_{x_j}^{x_{j+1}} \underbrace{(x_{j+1}-x)}_{F}\underbrace{(x-x_j)}_{G}\,dx = [PI]$$

$$= \frac{1}{h^2}\left[\cancel{(x_{j+1}-x)\frac{(x-x_j)^2}{2}}\Big|_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} -\frac{(x-x_j)^2}{2}\,dx\right]$$

$$= \frac{1}{6h^2}(x-x_j)^3\Big|_{x_j}^{x_{j+1}} = \frac{1}{6h^2}\cdot h^3 = \frac{1}{6}h$$

Obviously $\int_0^1 \varphi_i\,\varphi_j = 0$ for $|i-j| > 1$.

Thus the mass-matrix M is:

$$M = h\begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdots & & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\ & & & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix} = \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & \cdots & & 0 \\ 1 & 4 & 0 & & & \\ 0 & 1 & 4 & & & \\ & & & \ddots & & \\ 0 & \cdots & 0 & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}.$$